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**A REPORT ON AN EXTENSION OF ELASTICITY
THEORY TO INCLUDE GRANULAR MATERIALS**

in cooperation with the
Department of Commerce
Bureau of Public Roads

**TECHNICAL REPORT NO. 2
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A Report on
AN EXTENSION OF ELASTICITY THEORY TO INCLUDE
GRANULAR MATERIALS

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TECHNICAL REPORT NO. 2

For

RESEARCH PROJECT 2-8-62-27 (HPS-1-27)
"Distribution of Stresses in Layered Systems Composed
of Granular Materials"

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in cooperation with
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U. S. Bureau of Public Roads

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AN EXTENSION OF ELASTICITY THEORY TO INCLUDE GRANULAR MATERIALS

SYNOPSIS

In Technical Report No. 1 of this series there was presented a hypothesis describing in mathematical terms the deformation characteristics of granular materials. In the present report the differential equations of the theory of elasticity are extended to include materials that deform in accordance with the hypothesis.

The new equations resemble those of elasticity but are non-linear, and it appears that solutions of practical interest can be found only by the use of approximate numerical methods programmed for high speed computers.

1. INTRODUCTION

The first objective of Research Project 2-8-62-27, "The Distribution of Stress in Layered Systems Composed of Granular Materials," required the development and testing of a general hypothesis relating the stress to the deformation occurring at a point in a compacted mass of granular material as the result of a load applied to the boundary of the mass. The required hypothesis was presented in Technical Report No. 1 of this series, "A Mathematical Model Describing the Deformation Characteristics of Granular Materials," by Wayne A. Dunlap.

The second objective of the project calls for the development of a theory of stress distribution in layered systems based on the deformation hypothesis mentioned above. This report, the second of the series, is concerned with the development of the required theory.

Other objectives require that predictions of pavement behavior made from the theory be compared with actual behavior of Texas pavements and pavements included in the AASHO Road Test. The results of work now underway in accordance with these objectives will be the subject of later reports.

2. DEFORMATION CHARACTERISTICS OF GRANULAR MATERIALS

The Hypothesis described in Technical Report No. 1 may be stated as follows:

If at a point in a stressed mass of granular material two principal stresses are held constant while the third is increased, the ratio of this increase to the corresponding increase in principal strain is linearly dependent on the sum of the two stresses held constant.

As an aid to the mathematical statement of the hypothesis, we shall suppose that the three principal stresses act parallel to the axes of a cylindrical coordinate system (Figure 1). In accordance with the usual convention, tensile stresses will be considered positive in sign. In addition, to permit the use of differential calculus, we assume that the material is homogeneous. Under these conditions the hypothesis is represented mathematically by the three equations given below:

$$\frac{\partial \sigma_r}{\partial \epsilon_r} = K_2 - K_3 (\sigma_\theta + \sigma_z)$$

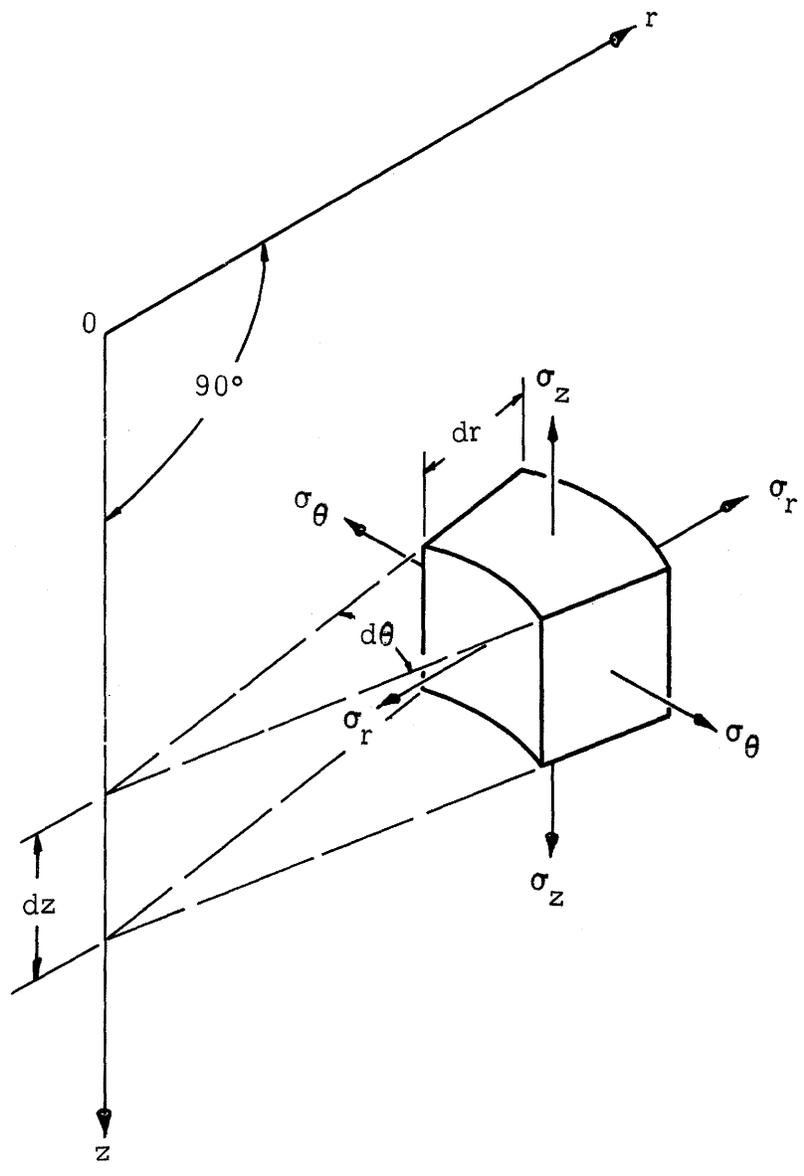


Figure 1: Coordinate system for Equations 1.

$$\frac{\partial \sigma_{\theta}}{\partial \epsilon_{\theta}} = K_2 - K_3 (\sigma_z + \sigma_r) \quad (1)$$

$$\frac{\partial \sigma_z}{\partial \epsilon_z} = K_2 - K_3 (\sigma_r + \sigma_{\theta})$$

where σ_r , σ_{θ} and σ_z are principal stresses acting parallel to the r, θ and z axes, respectively; and ϵ_r , ϵ_{θ} and ϵ_z are the corresponding principal strains. The symbols K_2 and K_3 represent material constants that depend on the unit weight, moisture content, graduation, rate of loading and other factors discussed in Technical Report No. 1. Evidence tending to confirm one of these equations was presented in that report, while the assumption of homogeneity leads to the other two.

It is of interest at this point to compare Equations 1 with corresponding equations from the theory of elasticity. The equations relating stress and strain in that theory are given below (1):

$$\begin{aligned} \sigma_r &= E\epsilon_r + \mu (\sigma_{\theta} + \sigma_z) \\ \sigma_{\theta} &= E\epsilon_{\theta} + \mu (\sigma_z + \sigma_r) \\ \sigma_z &= E\epsilon_z + \mu (\sigma_r + \sigma_{\theta}) \end{aligned} \quad (2)$$

where E is the modulus of elasticity and μ is Poisson's ratio.

By differentiating Equation 2, we find, for purely elastic materials:

$$\begin{aligned} \frac{\partial \sigma_r}{\partial \epsilon_r} &= E \\ \frac{\partial \sigma_{\theta}}{\partial \epsilon_{\theta}} &= E \\ \frac{\partial \sigma_z}{\partial \epsilon_z} &= E \end{aligned} \quad (3)$$

By comparing Equations 1 with Equations 3, it can be seen that if K_3 is set equal to zero Equations 1 correspond to elastic theory provided that K_2 is regarded as the modulus of elasticity.

In order to permit the integration of Equations 1, we shall assume that the integral of each is a function only of the variables shown in the differential form; for example, we assume that σ_r can be expressed as a function only of ϵ_r , σ_θ and σ_z as in the case of purely elastic materials. Then integration of each equation in turn yields the following:

$$\begin{aligned}\sigma_r &= K_2 \epsilon_r - K_3 (\sigma_\theta + \sigma_z) \epsilon_r + F (\sigma_\theta, \sigma_z) \\ \sigma_\theta &= K_2 \epsilon_\theta - K_3 (\sigma_z + \sigma_r) \epsilon_\theta + F (\sigma_z, \sigma_r) \\ \sigma_z &= K_2 \epsilon_z - K_3 (\sigma_r + \sigma_\theta) \epsilon_z + F (\sigma_r, \sigma_\theta)\end{aligned}\tag{4}$$

where the functions, F , arise as a result of the integration. Because of the assumption of homogeneity, the functions F have the same general form in all three equations.

Inasmuch as setting K_3 equal to zero in Equations 1 yields a set of equations consistent with elasticity theory, it is assumed - in the absence of experimental evidence to the contrary - that setting K_3 equal to zero in the integrals of those equations should also yield results consistent with elasticity theory. With K_3 zero, Equations 4 reduce to the following:

$$\begin{aligned}\sigma_r &= K_2 \epsilon_r + F (\sigma_\theta, \sigma_z) \\ \sigma_\theta &= K_2 \epsilon_\theta + F (\sigma_z, \sigma_r) \\ \sigma_z &= K_2 \epsilon_z + F (\sigma_r, \sigma_\theta)\end{aligned}\tag{5}$$

By comparing Equations 5 with the corresponding expressions from elasticity theory (Equations 2), it can be seen that K_2 again corresponds to E , while the functions F are evidently the following:

$$\begin{aligned}F (\sigma_\theta, \sigma_z) &= K_1 (\sigma_\theta + \sigma_z) \\ F (\sigma_z, \sigma_r) &= K_1 (\sigma_z + \sigma_r) \\ F (\sigma_r, \sigma_\theta) &= K_1 (\sigma_r + \sigma_\theta)\end{aligned}\tag{6}$$

where K_1 is analogous to Poisson's ratio.

By substituting Equations 6 in Equations 5 and rearranging the terms in each, we obtain the mathematical expression of the deformation hypothesis given below:

$$\begin{aligned}\epsilon_r &= \frac{\sigma_r - K_1 (\sigma_\theta + \sigma_z)}{K_2 - K_3 (\sigma_\theta + \sigma_z)} \\ \epsilon_\theta &= \frac{\sigma_\theta - K_1 (\sigma_z + \sigma_r)}{K_2 - K_3 (\sigma_z + \sigma_r)} \\ \epsilon_z &= \frac{\sigma_z - K_1 (\sigma_r + \sigma_\theta)}{K_2 - K_3 (\sigma_r + \sigma_\theta)}\end{aligned}\tag{7}$$

Up to this point the normal stresses, σ_r , σ_θ , and σ_z , have been defined as principal stresses because in the laboratory experiments used in testing the deformation hypothesis no shear stresses were applied to the triaxial test specimens. In order to make Equations 7 generally applicable to problems of stress distribution, we must assume - as in the theory of elasticity - that shears produce only angular distortion and have no effect on normal strains.

To summarize: Equations 7 are regarded as an extension of the deformation equations of elasticity to include the granular materials used in the construction of flexible pavements. The dimensionless constant, K_1 , is analogous to Poisson's ratio. The constant, K_2 , corresponds to the modulus of elasticity when the constant K_3 is zero. The latter is a dimensionless constant that has no counterpart in elasticity, but represents a characteristic property of granular materials. It is the existence of K_3 that warrants the development of a special theory of stresses in granular materials. Laboratory determined values of this constant have been found to range as high as 1500 (See Report No. 1).

3. THE DIFFERENTIAL EQUATIONS OF ELASTICITY

EXTENDED TO INCLUDE GRANULAR MATERIALS

The Layered System Problem: The problem of determining stresses and displacements in layered systems composed of purely elastic materials was first solved by Professor Donald M. Burmister (2). It is an objective of this research project to extend the theory of layered systems to include granular materials that behave in accordance with the hypothesis treated in the preceding section.

Figure 2 illustrates the layered system problem as it was treated by Burmister and others (3), (4), (5). As indicated in the figures, the stressed body consists of two or more layers assumed to extend horizontally an

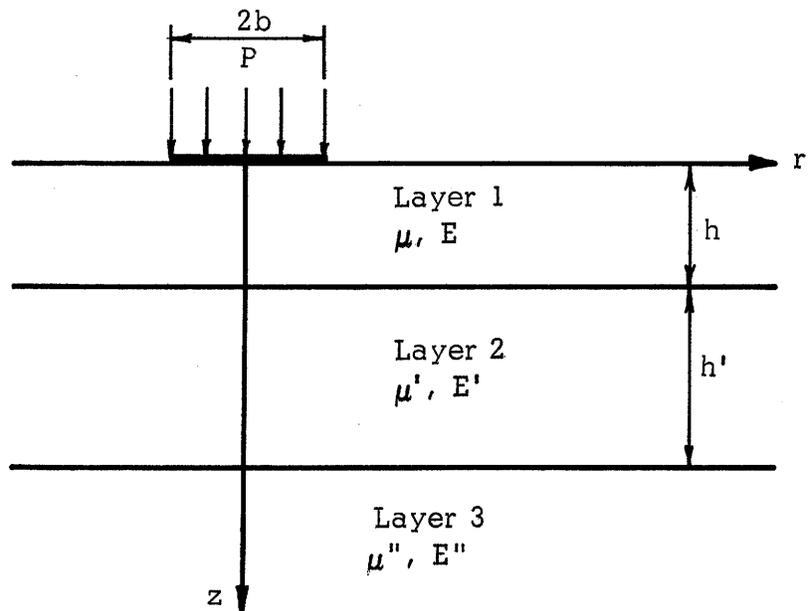


Figure 2: Parameters for a three-layer system of perfectly elastic materials.

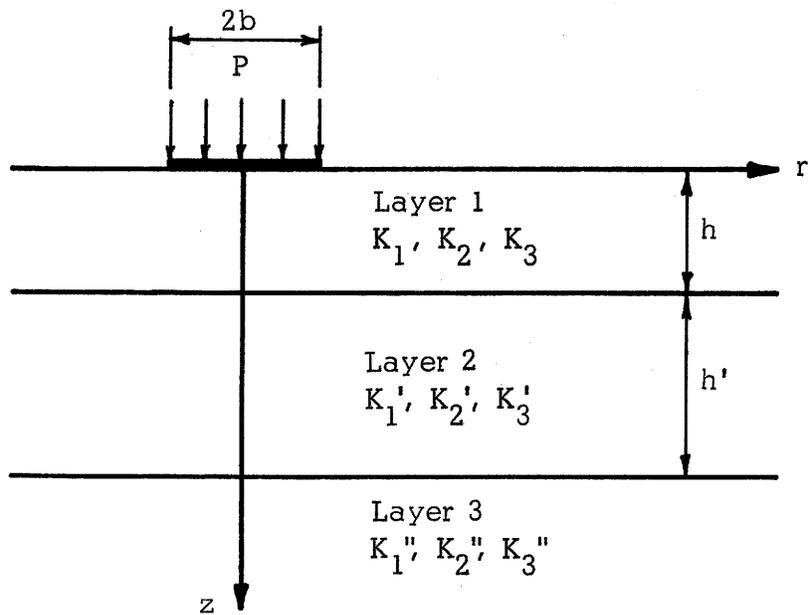


Figure 3: Parameters for a three-layer system of granular materials.

indefinite distance. Each layer is characterized by a modulus of elasticity, E , a Poisson's ratio, μ , and a thickness, h , except that in the case of the deepest layer the thickness is assumed to be infinite. The system is loaded by a uniform vertical pressure, P , distributed over a circular area of radius b .

An origin of cylindrical coordinate is taken at the center of the loaded area. Because of the symmetry of the layered system about the z axis the only non-zero shear stress is τ_{rz} . Thus σ_θ is a principal stress while σ_r and σ_z in general are not.

The Elastic Equations Applied to the Problem: Fundamentally there are six functions of r and z to be determined: the stresses σ_r , σ_θ , σ_z and τ_{rz} ; the displacement u parallel to the r -axis; the displacement w parallel to the z -axis. These functions must conform to specified conditions on the boundaries of each layer and they must satisfy the six equations given below:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

$$\frac{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}}{\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z}} = \frac{2\tau_{rz}}{\sigma_r - \sigma_z} \quad (8)$$

$$\frac{\partial u}{\partial r} = \frac{1}{E} \left[\sigma_r - \mu (\sigma_\theta + \sigma_z) \right]$$

$$\frac{u}{r} = \frac{1}{E} \left[\sigma_\theta - \mu (\sigma_z + \sigma_r) \right]$$

$$\frac{\partial w}{\partial z} = \frac{1}{E} \left[\sigma_z - \mu (\sigma_r + \sigma_\theta) \right]$$

The boundary conditions are stated below:

1. Within the loaded area ($z=0, r < b$): $\sigma_z = -P, \tau_{rz} = 0$.
2. On the upper surface, outside the loaded area ($z=0, r > b$): $\sigma_z = \tau_{rz} = 0$.
3. On any interface: $\sigma_z' = \sigma_z, \tau_{rz}' = \tau_{rz}, u' = u, w' = w$.

(The primed quantities refer to stresses and displacements in the lower layer, the unprimed quantities to those in the upper layer.)

4. At $r = \text{infinity}$ and $z = \text{infinity}$ all stresses and displacements are zero.

The first two Equations 8 insure that conditions of static equilibrium are satisfied throughout the layered system.

The third equation expresses the special relationship between shear and normal stresses and strains in the r - z plane resulting from the fact that τ_{rz} is the only non-zero shear stress. In terms of strains, this equation is

$$\frac{\gamma_{rz}}{\epsilon_r - \epsilon_z} = \frac{2\tau_{rz}}{\sigma_r - \sigma_z} \quad (9)$$

where γ_{rz} is the shear strain.* When the strains are replaced by equivalent expressions involving the displacements u and w , Equation 9 takes the form given as the third of Equations 8. The relations between strains and displacements are shown below:

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} \\ \epsilon_\theta &= \frac{u}{r} \\ \epsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \quad (10)$$

*Equation 9 reduces to $\tau_{rz}/\gamma_{rz} = \text{a constant}$ for purely elastic materials ($K_3=0$), but in the more general case ($K_3 \neq 0$), τ_{rz}/γ_{rz} is a function of the normal stresses. (See Equations 11.)

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1. Within the loaded area ($z=0, r<b$): $\sigma_z = -P, \tau_{rz} = 0$.
2. On the upper surface, outside the loaded area ($z=0, r>b$): $\sigma_z = \tau_{rz} = 0$.
3. On any interface: $\sigma_z' = \sigma_z, \tau_{rz}' = \tau_{rz}, u' = u, w' = w$.

(The primed quantities refer to stresses and displacements in the lower layer, the unprimed quantities to those in the upper layer.)

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The last three of Equations 8 were obtained from Equations 2 by replacing the strains ϵ_r , ϵ_θ and ϵ_z by corresponding expressions involving the displacements (Equations 10).

Equations 8 may be confirmed by reference to a standard text on the theory of elasticity (1).

The boundary conditions are those used by Professor Burmister for the condition of perfect continuity at the interface (as opposed to an interface permitting the frictionless sliding of one layer on the other).

Equations of Elasticity Extended to Layered Systems Composed of Granular Materials: A proposed system of six equations applying to a layered system composed of granular materials (Figure 3) is given below:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

$$\frac{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}}{\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z}} = \frac{2\tau_{rz}}{\sigma_r - \sigma_z} \quad (11)$$

$$\frac{\partial u}{\partial r} = \frac{\sigma_r - K_1 (\sigma_\theta + \sigma_z)}{K_2 - K_3 (\sigma_\theta + \sigma_z)}$$

$$\frac{u}{r} = \frac{\sigma_\theta - K_1 (\sigma_z + \sigma_r)}{K_2 - K_3 (\sigma_z + \sigma_r)}$$

$$\frac{\partial w}{\partial z} = \frac{\sigma_z - K_1 (\sigma_r + \sigma_\theta)}{K_2 - K_3 (\sigma_r + \sigma_\theta)}$$

Only the last three equations, which state the deformation hypothesis for granular materials, differ from Equations 8 for purely elastic materials. The first three, which are based on considerations of static equilibrium and geometry, are taken to be independent of the deformation characteristics of the material.

The boundary conditions are not affected by the deformation characteristics of the material and are therefore the same as those already given for purely elastic materials.

This completes the development of the theory for the distribution of stress in layered systems composed of granular materials. The problem of determining the stresses consists of finding six functions of the coordinates, r and z , that satisfy Equations 11 and the conditions specified on the boundaries.

4. POSSIBLE METHODS OF SOLUTION

The nonlinear character of Equations 11 appears to preclude the possibility that an exact solution of the layered system problem for granular materials can be found. However, approximate solutions are believed to be possible using numerical techniques programmed for high speed computers.

One such technique - with a host of variations - involves the substitution of finite difference equations for differential equations, and the solution of the latter by iterative processes. Some of the variations on this technique have been under investigation but convergence to a solution has not yet been achieved.

Another approach to obtaining a numerical solution to the above equations is being considered. This method, like the difference method, assumes a set of starting values. These starting values make it possible to approximate each of the six unknown functions by a truncated Fourier series of orthogonal polynomials. Corrections can then be computed by successive substitution into the six equations and their associated boundary conditions. Although this method has not been thoroughly tested, there is some reason to hope that it will result in the suppression of certain localized conditions which sometimes render finite difference methods numerically unstable.

5. CONCLUSIONS

From the experimental data presented in Technical Report No. 1 of this series, and from the equations presented in this report, it is believed that the following conclusions are warranted:

1. Laboratory tests on compacted specimens of granular materials have tended to confirm the existence of a dimensionless deformation constant, K_3 , which has no counterpart in the theory of elasticity but is characteristic of granular masses (Equations 1).

2. The existence of K_3 has required the development of new differential equations governing the distribution of stress in layered systems composed of granular materials (Equations 11).

3. Since the new equations reduce to the equations of elasticity when K_3 is zero, they are not limited to systems with all layers composed of granular materials but apply equally well to mixed systems in which some of the layers are purely elastic and some are granular.

4. While the new equations are nonlinear, and therefore difficult to solve, it is likely that solutions of practical interest can be found by use of numerical techniques programmed for high speed computers. Efforts toward that end should be continued as a part of the work in this project.

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