Safety Provisions for Support Structures on Overhead Sign Bridges: A Study of Buckling Stress Formulas

1969

Texas Transportation Institute

TEXAS TRANSPORTATION INSTITUTE
TEXAS A&M RESEARCH FOUNDATION

TECHNICAL MEMORANDUM 605-3

A STUDY OF BUCKLING STRESS FORMULAS

SAFETY PROVISIONS FOR SUPPORT STRUCTURES

ON OVERHEAD SIGN BRIDGES

By

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Prepared for the Department of Transportation, Federal Highway Administration, Bureau of Public Roads, under Contract No. FH-11-7032. Financed with 1 1/2 percent HPR funds made available by the several States and administered by the Bureau of Public Roads.

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December 1969

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INTRODUCTION

This report deals with the problems of the critical stress design requirements for the supports of overhead sign bridges. The following discussion will serve to illustrate one point, namely that the present AASHO[1] requirement for critical stress on overhead sign bridges is overly conservative. This is due primarily to the fact that the type of structure and loading upon which the critical stress requirement is based does not represent an actual sign bridge support. The conservatism leads to much heavier column sections than are actually required. This is not an important factor in the design of stationary structures such as bridges, buildings or present-day overhead sign bridges. However, it becomes a very important factor in the design of break-away supports. The weight of the supports may be reduced appreciably while maintaining structural stability. The effects upon the occupants in a vehicle colliding with a lighter break-away support could be significant. In fact, it may mean the difference between life and death.

The *AASHO Specifications for the Design and Construction of Structural Supports for Highway Signs*,[1] (published by the American Association of State Highway Officials) uses Formula 20 of the United States Steel *Design Manual for High Strength Steels*[2] as its criterion for buckling stresses. This formula is:

$$f_{cr} = \frac{\Pi^2 E}{2(\frac{L}{d})^2} \sqrt{\left(\frac{I_y}{2I_x}\right)^2 + \frac{KI_y}{2(1+\mu)\ I_x^{\,2}} \left(\frac{L}{\Pi d}\right)^2} \qquad (1)$$

This formula was derived by Winter[8,10].

If I_x is several times I_y, this reduces to Formula 21 (derived by deVries[9]).

$$f_{cr} = \frac{18.83 \times 10^6}{\frac{Ld}{bt}} \tag{2}$$

These equations are derived from the lateral buckling of beams in pure bending (Figure 1).

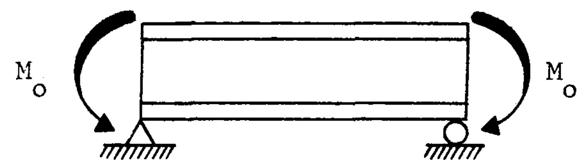

Figure 1

S. Timoshenko gives a formula similar to the USS Formula 20 in his text: *Theory of Elastic Stability*.[5] Timoshenko also presents an equation for the lateral buckling of a cantilever beam under lateral end load as (Figure 2):

$$P_{cr} = \frac{4.013\sqrt{EI_y GK}}{\left(1 - \sqrt{\frac{EI_y d^2}{4GKL^2}}\right) L^2} \tag{3}$$

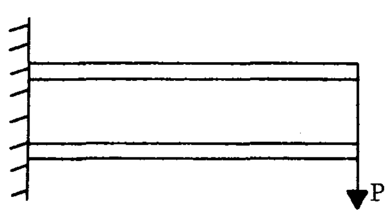

Figure 2

The theoretical buckling load for a perfectly straight slender member under axial compression is the famous Euler Buckling Load (See Figure 3).

$$P_E = \frac{\Pi^2 E I_y}{(K_y L)^2} \tag{4}$$

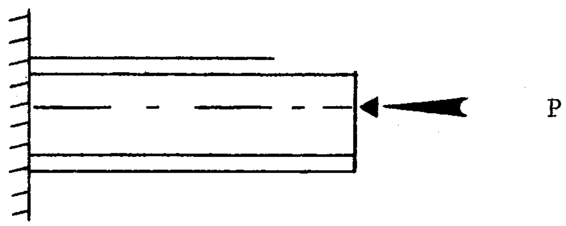

Figure 3

Another expression for determining critical axial loads on columns with a small amount of eccentricity is the Secant Formula (See Figure 4):

$$f_{cr} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \operatorname{Sec}\left(\frac{L}{2r} \sqrt{\frac{P}{AE}}\right)\right] \quad 5)$$

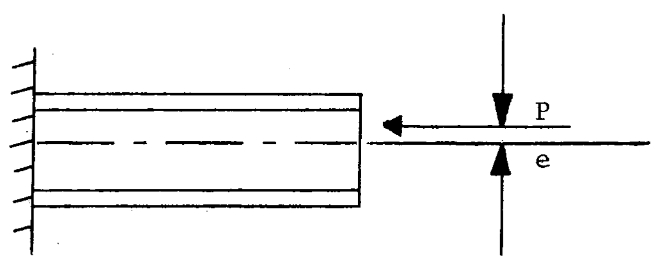

Figure 4

The loading on an overhead sign bridge support is actually a combination of all of these types of loads (Figure 5). However, no formulas for calculating critical stress exist for the general loading case.

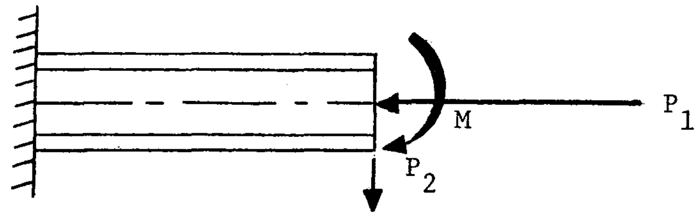

Figure 5

The theoretical buckling load may never be reached if the stresses exceed the yield point before large deflections occur; a common occurrence with the Euler Buckling Formula. It may be useful to determine under what conditions to use yield-point stress or buckling stress as the basis for design.

If individual requirements for Ld/bt, K_yL/r_y and L/r_{yf} are met, the member should fail due to yielding rather than elastic buckling, irregardless of the loads placed on the member. However, this leads to a very conservative design in most cases. In addition, since no safety factors have been considered, the member sizes may be unconservative in limiting cases. It seems logical, therefore, to arrive at a consistent Safety Factor to apply to these formulae and then devise some sort of interaction formula to check the combined stresses, such as AISC Formula 7a:

$$\frac{f_a}{F_a} + \frac{C_m F_b}{\left(1 - \frac{f_a}{F'_e}\right) F_b} \leq 1.0 \qquad (6)$$

possibly in the form

$$\frac{f_a}{F_a} + \frac{f_{bC}}{F_{bC}} + \frac{f_{bM}}{F_{bM}} \leqq 1.0 \qquad (7)$$

where the subscript "a" refers to axial, "bC" refers to cantilever type bending, and "bM" refers to bending due to moment loading.

The first term includes effects of K_yL/r_y (See Figure 6a and 6d), the second term includes effects of I_y and K_y (See Figures 6b and 6e), and the third term includes effects of Ld/bt (See Figures 6c and 6f); i.e., the terms include effects of axial load, cantilever bending, and simple bending, respectively.

The work done by Krefeld and correlated with the present computer solution, indicates that: (1) the AASHO Specifications for buckling stress of sign-bridge supports are not consistent with the loading and boundary conditions present in the cantilever column supports, and (2) that a formula can be devised to express the critical stress for this case in terms of cross section properties and material properties.

LIMITING VALUES OF BUCKLING PARAMETERS

It is possible to arrive at limiting values of Ld/bt, L/r_{yf}, and K_yL/r_y which insure failure by yielding rather than failure by elastic buckling.

In *Structural Steel Design*[12:200] two equations (7.28 and 7.29) are given for critical stresses for shallow, thick-walled beams and deep, thin-walled beams, respectively.

$$(7.28) \quad F_{cr} = \frac{0.65E}{\frac{Ld}{bt}} \tag{8}$$

$$(7.29) \quad F_{cr} = \frac{\Pi^2 E}{\left(\frac{K_y L}{r_{yf}}\right)^2} \tag{9}$$

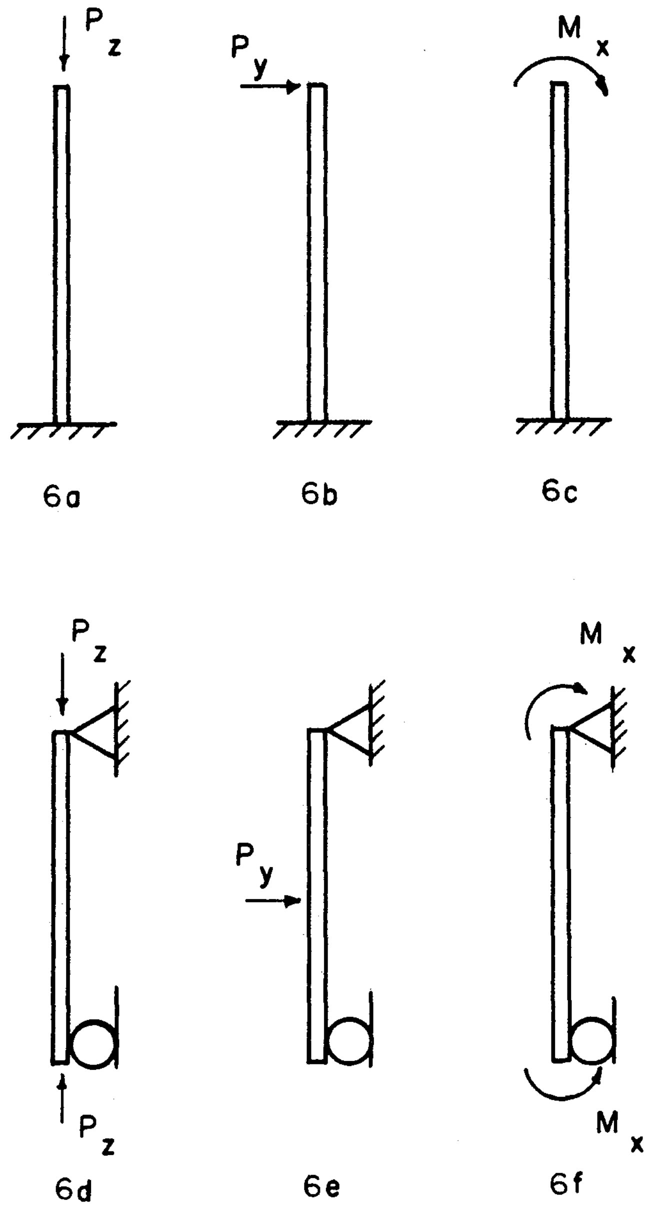

FIGURE 6. LOADING AND END FIXITY CONDITION

where:

$$r_{yf} = \sqrt{\frac{I_{yf}}{A_f + \frac{1}{6} A_W}}$$

if F_{cr} is set equal to F_y, these yield:

$$\frac{Ld}{bt} = \frac{0.65E}{F_y} \tag{11}$$

$$\frac{K_y L}{r_{yf}} = \Pi \frac{\sqrt{E}}{F_y} \tag{12}$$

for $E = 29 \times 10^6$ psi and $F_y = 36{,}000$ psi, these yield limiting values of $Ld/bt = 5250$ and $K_y L/r_f = 89.5$.

The equations above were developed for a simple beam under pure bending (See Figure 6f). Other similar equations are given by deVries[9], Winter[8,10] and McGuire[11]. For axial loading, Euler's critical stress formula:

$$F_{cr} = \frac{\Pi^2 E}{\left(\frac{K_y L}{r_y}\right)^2} \tag{13}$$

yields maximum values of $K_y L/r_y$ of

$$\frac{K_y L}{r_y} = \Pi \frac{\sqrt{E}}{F_y} \tag{14}$$

which is the same form as Equation (12) above except that Equation (12) refers to buckling of the compression flange whereas Equation (13) refers to buckling of the entire section. If $E = 29 \times 10^6$ psi and

$F_y = 36{,}000$ psi, the limiting value for K_yL/r_y is 89.5.

The critical point load on the end of a cantilever[5] (Figure 6b) is:

$$P_{cr} = \frac{4.013\sqrt{EI_y GK}}{\left(1 - \frac{d}{2L}\sqrt{\frac{EI_y}{GK}}\right)L^2} \tag{15}$$

where, from now on, the K refers to the torsional rigidity and the radii of gyration will be subscripted to denote axis of bending. The critical stress at the support is:

$$F_{cr} = \frac{P_{cr}\, L(\frac{d}{2})}{I_x} \tag{16}$$

which yields a limiting value of

$$\frac{I_y \cdot K}{(K_y)^4} = \frac{1}{EG\,(\frac{d}{2L})^2}\left[\frac{F_y}{\frac{4.013\,(K_x)^2}{I_x} + \frac{(K_y)^2}{GK}}\right]^2 \tag{17}$$

this can be rewritten in the Ld/bt form[10] to yield a limiting value of

$$\frac{Ld}{bt} = \frac{4}{3}\,(\frac{K_x}{K_y})^2\,\frac{\sqrt{EG}}{F_y} + \frac{1}{4}\,(\frac{d}{t})^2\sqrt{\frac{E}{G}} \tag{18}$$

If $K_x = K_y = 2$, $E = 29 \times 10^6$ psi, $G = 11 \times 10^6$ psi, $F_y = 36{,}000$ psi and $d/t = 20$, then $Ld/bt = 820$.

The AASHO requirement for critical stress (Formula 21 of *Design Manual for High Strength Steels*[2])

$$F_{cr} = \frac{(18.83)\,10^6}{\frac{Ld}{bt}} \tag{19}$$

yielding limiting values for Ld/bt of

$$\frac{Ld}{bt} = \frac{(18.83)\ 10^6}{F_y} \tag{20}$$

which is equivalent to Equation (11) above.

The critical bending moment causing lateral buckling of a simple beam is given by Timoshenko[5] as:

$$(M_o)_{cr} = \gamma_2 \frac{\sqrt{EI_y GK}}{L} \tag{21}$$

where the smallest value of $\gamma_2 = \Pi$.

The critical load for lateral buckling of a simple beam due to a lateral point load in the center[5] is:

$$P_{cr} = \gamma_2 \frac{\sqrt{EI_y GK}}{L^2} \tag{22}$$

where $\gamma_2 \doteq 15.0$ for concentrated load at the middle of the upper flange. For a uniformily distributed load along the upper flange of a simple beam $\gamma_2 = 26.0$.

These loadings result in higher limiting values of Ld/bt than the pure bending case. This is why the pure bending case is used in the determination of the critical stress. However, as will be shown, this is very conservative in the case of a sign bridge support.

Timoshenko[5] gives a critical load for a cantilever loaded at the end point in the same form as Equation (22).

$$P_{cr} = \frac{\gamma_1}{L^2}\sqrt{EI_y GK} \tag{23}$$

in which the lowest value of γ_1 is 4.013. Timoshenko[5] also investigated the case of a cross section for which the moment of inertia I_y varies as

$$I_y = I_{yo}\left(1 - \frac{z}{e}\right)^n \tag{24}$$

The γ_1 term for various values of n is shown in the following table.

n	0	1/4	1/2	3/4	1
Uniform load	12.85	12.05	11.24	10.43	9.62
Concentrated load at free end	4.013	3.614	3.214	2.811	2.405

Table 1. Values of γ_1 as a Function of Taper

This follows the same trend as the reduction factors for tapered beams, $R = \frac{7 + \alpha}{5 + 3\alpha}$ and $R = \frac{4 + \alpha}{3 + 2\alpha}$, determined by Krefeld, et al.[4]. ($\alpha = \frac{Z_o}{Z_1} \times \left(\frac{b_o}{b_1}\frac{d_1}{d_o}\right)^{3/2}$ is a taper parameter, $\alpha \geq 1$). Krefeld's work will be discussed later.

The standard Euler buckling load rewritten in this manner yields a maximum value of $K_y L/r_y$.

$$\frac{K_y L}{r_y} = \Pi \sqrt{\frac{E}{F_y}} \qquad (14)$$

which for A36 Steel is

$$\frac{K_y L}{r_y} \leqq 89.5 \qquad (25)$$

and for higher strength steel with F_y of 100,000 psi

$$\frac{K_y L}{r_y} \leqq 58.3 \qquad (26)$$

This example illustrates the paradox associated with buckling and high strength materials. The fact that the material is stronger does not mean that it is a more economical material to use when buckling is a problem. In many cases, the high-strength steels cannot be used to their full capacity because the critical stress will be lower than the allowable stress. In these cases, a lower strength steel is adequate and more economical. This is not meant to say that larger values of $K_y L/r_y$, $K_y L/r_{fy}$ or Ld/bt cannot be used than are obtained from the preceeding formulas. It just means that if larger values are used, the possibility of buckling at stresses below the yield point exists and thus the allowable stresses must be reduced.

CANTILEVER BEAM-COLUMNS WITH END RESTRAINT

The text: *Theory of Elastic Stability*, by S. P. Timoshenko[5] gives the following formula for critical buckling load of a cantilever beam under lateral end load (See Figure 6b).

$$P_{cr} = \frac{4.013\sqrt{EI_yGK}}{\left(1 - \sqrt{\frac{EI_y\ d^2}{4GKL^2}}\right)^2 L^2}\left(1 - \frac{a}{L}\sqrt{\frac{EI_y}{GK}}\right) \tag{27}$$

The term $\left(1 - \frac{a}{L}\sqrt{\frac{EI_y}{GK}}\right)$ is a correction factor to account for the load being applied at points other than the centroid. If the load is applied to the upper flange, a is $\frac{d}{2}$ and the equation becomes:

$$P_{cr} = \frac{4.013\sqrt{EI_yGK}}{\left(1 - \frac{d}{2L}\sqrt{\frac{EI_y}{GK}}\right)L^2} \cdot \tag{28}$$

The maximum bending stress would occur at the fixed end:

$$f_{cr} = \frac{Mc}{I} = \frac{P_{cr}\cdot L\cdot \frac{d}{2}}{I_x} = \frac{4.013}{I_x}\left(\frac{d}{2L}\right)\frac{\sqrt{EI_yGK}}{\left(1 - \frac{d}{2L}\sqrt{\frac{EI_y}{GK}}\right)} \tag{29}$$

If the effective lengths and radii of gyration are taken into account, the formula becomes:

$$f_{cr} = 4.013\left(\frac{d}{2L}\right)\left(\frac{K_x^2}{I_x}\right)\frac{\sqrt{EG\sqrt{\frac{I_y}{K_y^2}\cdot\frac{K}{K_z^2}}}}{\left(1 - \frac{d}{2L}\sqrt{\frac{E}{G}\ \frac{I_y}{K_y^2}\cdot\frac{K_z^2}{K}}\right)} \tag{30}$$

where K_x, K_y, and K_z were each 2.0 in the original derivation. In terms of radii of gyration, this becomes:

$$f_{cr} = 4.013 \left(\frac{d}{2L}\right)\left(\frac{K_x L}{r_x}\right)^2 \frac{\sqrt{EG}\left(\frac{r_y}{K_y L}\right)\left(\frac{r_z}{K_z L}\right)}{\left(1 - \frac{d}{2L}\sqrt{\frac{E}{G}} \cdot \frac{r_y}{r_z}\right)} \tag{31}$$

The term r_z might be termed the torsional radius of gyration. It is related to the usual Ld/bt term, but the connection is somewhat vague. This expression clearly shows that this type of buckling is a combination of weak axis bending and torsion. The effective length factor, K_z, refers to the effective length of the compression flange during buckling and is therefore often the same as K_y. Later K_z will be replaced by K_y.

The effective length, $K_y L$, can be obtained from Figure C1.8.3 of the *AISC Manual of Steel Construction*[3] using a G for the partially restrained end obtained in the following manner.

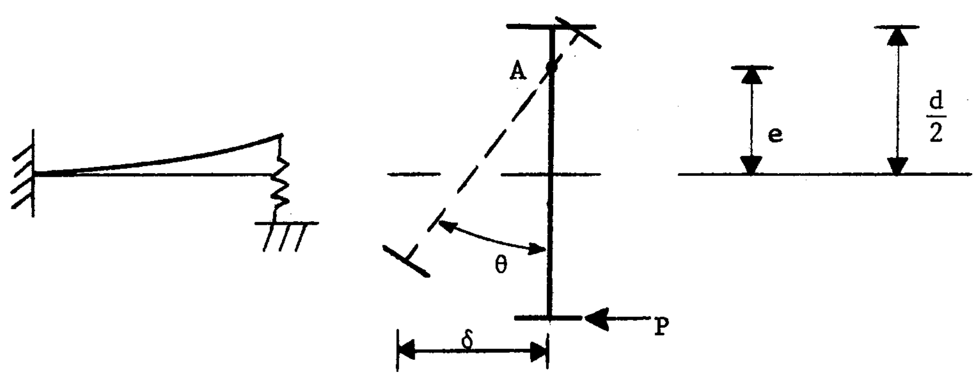

Figure 7

The point "A" denotes the point of lateral support.

$$\delta_P = \frac{PL^3}{3E\left(\frac{I_y}{2}\right)} \tag{32}$$

$$\delta_T = \theta\left(\frac{d}{2} + e\right) = \frac{TL}{GK}\left(\frac{d}{2} + e\right) = \frac{PL\left(\frac{d}{2} + e\right)^2}{GK} \tag{33}$$

where δ_P is the lateral displacement of the compression flange due to a fictitious lateral load, P, and δ_T is the lateral displacement resulting in a resisting torque, T.

$$G_B = \frac{\delta_T}{\delta_P} = \frac{P\left(\frac{d}{2} + e\right)^2 L}{GK} \cdot \frac{3\ EI_y}{2\ PL^3} \tag{34}$$

$$G_B = \frac{3}{8}\ \frac{EI_y}{GK}\left(\frac{d}{L}\right)^2 \left(1 + \frac{2e}{d}\right)^2 \tag{35}$$

Let us now compare Equation (30) with the test results from Krefeld, et al.[4] A comparison of Equation (30) with Krefeld's results is shown on Figures 8 and 9. Consider specimen number T12 with L = 12 feet.

$b = 3.99$ in.	$d = 12$ in.
$t_f = 0.329$ in.	$t_w = 0.270$ in.
$\frac{Ld}{bt} = 1322$	$P_{cr} = 9{,}920$ lb.
$f_b = 69{,}600$ psi	$F_y = 95{,}250$ psi
$I_x = 128.5$ in.^{4}	
$I_y = 3.5$ in.^{4}	
$K = 0.174$ in.^{4}	

The end point brace at $e = d/2$ furnishes a constraint to bending about the weak axis and also restrains the rotation somewhat. Let us assume the following values:

$$E = 29 \times 10^6 \text{ psi}$$

$$K_x = 2.0$$

$$G = 11 \times 10^6 \text{ psi}$$

From Equation (35)

$$G_B = \frac{3}{8} (2.6) \left(\frac{3.5}{0.174}\right) \left(\frac{12}{144}\right)^2 (2)^2 = 0.55$$

$G_A = 0$

Entering Figure C1.8.3 of the *AISC Manual of Steel Construction*[3] with the above G values yields:

$$K_y = 1.1$$

Substituting the values listed above in the critical stress formula (Equation 30), one obtains:

$$f_{cr} = 86{,}500 \text{ psi}$$

This is somewhat greater than the test result of $f_b = 69{,}600$ psi. This is to be expected, since theoretical buckling analyses overestimate the buckling loads.

Consider another Krefeld beam, specimen number 12 JR A - 1.

$$I_x = 71.929 \doteq 72; \quad d = 12 \text{ in.}$$

$$I_y = 0.998 \doteq 1; \quad L = 192 \text{ in.}$$

$$K = 0.045$$

$$(f_b)_{cr} = 19{,}824 \text{ psi}$$

Substituting these values into Equation (35) for G_B yields

$$G_B = 0.34$$

$$G_A = 0$$

Entering Figure C1.8.3 of the *AISC Manual of Steel Construction*[3] with the above values yields:

$$K_y = 1.05$$

Using Equation (30) for the theoretical critical stress yields:

$$f_{cr} = 30{,}800 \text{ psi}$$

The test results gave $f_b = 19{,}824$ psi. The deviation between test results and theoretical calculations increase as the members become lighter. It seems logical, therefore, to incorporate the term I_x/I_y into the factor of safety to be applied to the theoretical stress. If we assume a factor of safety in the form,

$$F.S. = 2 \left[1 + \left(\frac{1}{100} \frac{I_x}{I_y} \right)^2 \right] \tag{36}$$

We arrive at a factor of safety applied to the theoretical stress which is always greater than 2.0 and when applied to the test results, ranges from 1.67 for low ratios of I_x/I_y to 2.0 for the higher values of I_x/I_y.

The allowable stress F_{bC} to use in the interaction formula would then be:

$$F_{bC} = \frac{4.013}{F.S.}\left(\frac{d}{2L}\right)\left(\frac{K_x^2}{I_x K_y^2}\right)\frac{\sqrt{EG}\sqrt{I_y K}}{\left(1 - \frac{d}{2L}\sqrt{\frac{E}{G}\frac{I_y}{K}}\right)} \tag{37}$$

If the web of the member contributes very little to I_x, I_y and K, the following approximate expressions can be used[10].

$$I_x \doteq 2\ bt\left(\frac{d}{2}\right)^2, \qquad I_y \doteq \frac{2tb^3}{12}, \qquad K \doteq \frac{2}{3}\ bt^3. \tag{38}$$

Substitution of these terms into Equation (37) (assuming $K_z = K_y$) yields:

$$F_{bC} = \frac{4}{3(F.S.)}\left(\frac{K_x}{K_y}\right)^2 \frac{\sqrt{EG}}{\left(\frac{Ld}{bt}\right)} \frac{1}{\left(1 - \frac{1}{4}\frac{\left(\frac{d}{t}\right)^2}{\left(\frac{Ld}{bt}\right)}\sqrt{\frac{E}{G}}\right)} \tag{39}$$

This equation yields allowable stress values which are usually a few percent higher than Equation (37).

COMPARISON OF CRITICAL STRESS FORMULAS

The following graphs compare various critical stress formulas as described in Table 2. In each case, it is assumed that the stresses are below the yield point. These curves are results from a representative

Table 2 Critical Stress Cases

CASE	DESCRIPTION	SOURCE
1	Euler buckling stress -- simple column, $K_y = 1.0$	Timoshenko,[5] p. 51
2	Euler buckling stress -- cantilever, K = 2.0	Timoshenko,[5] p. 51
3.	Allowable axial stress	*AISC Steel Manual*[3] Formulas 1 and 2
4	Lateral buckling of simple beam in pure bending	*USS Design Manual*[2] Formulas 21 and 22 (See also references 8, 9, and 10.)
5	Lateral buckling of cantilever beams under end load	Timoshenko,[5] p. 258
6	Allowable stress formulas based on lateral buckling	*AISC Steel Manual*[3] Formulas 4 and 5
7	Lateral buckling of restrained cantilever (modified Case 5)	Equations 30 and 37
8	Lateral buckling of restrained cantilever (modified Eq. 30)	Equation 39
9	Test data	Krefeld, et al.[4]

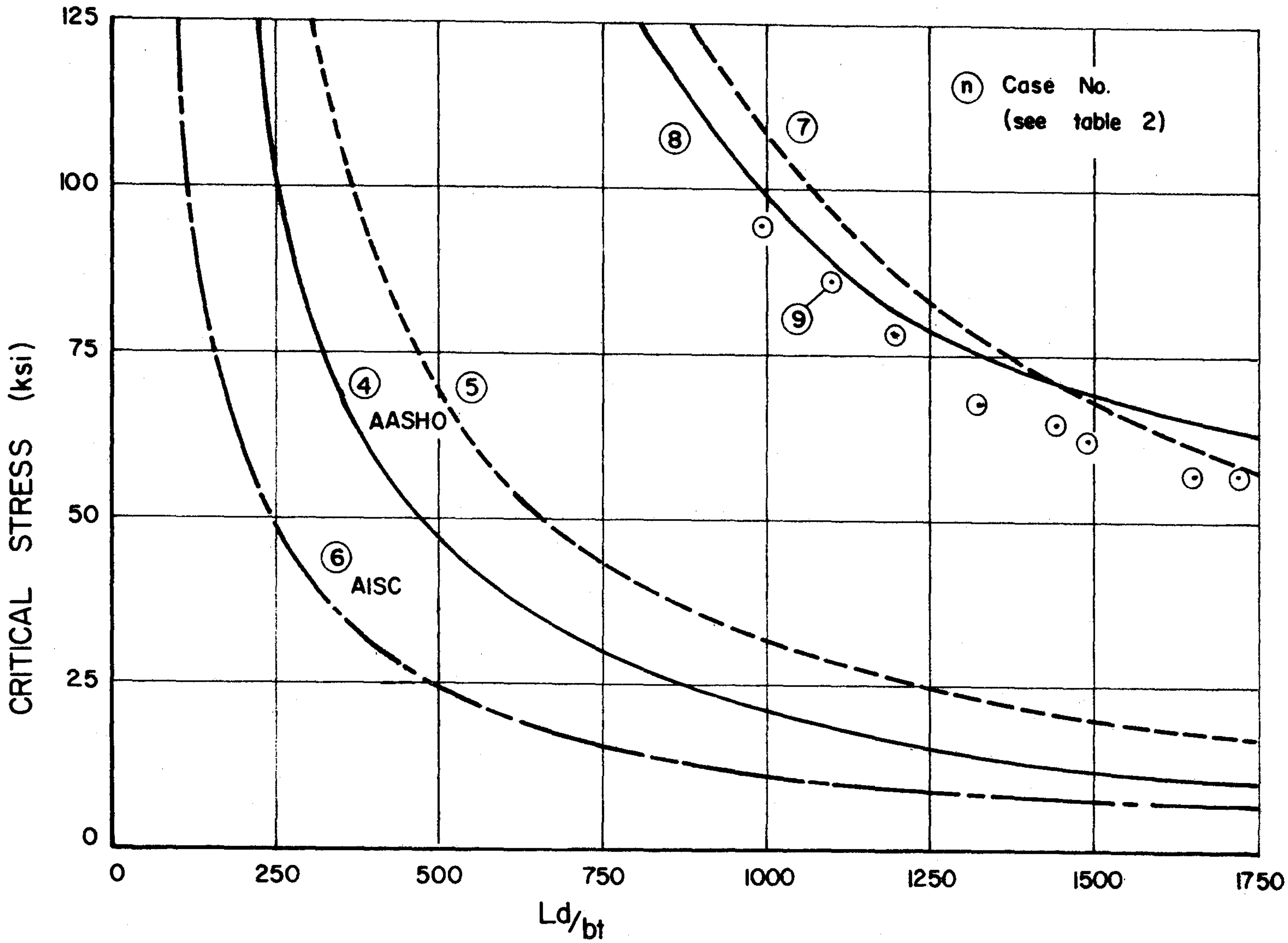

FIGURE 8. COMPARISON OF CRITICAL STRESSES vs. Ld/bt FROM VARIOUS THEORETICAL FORMULAS AND TEST RESULTS

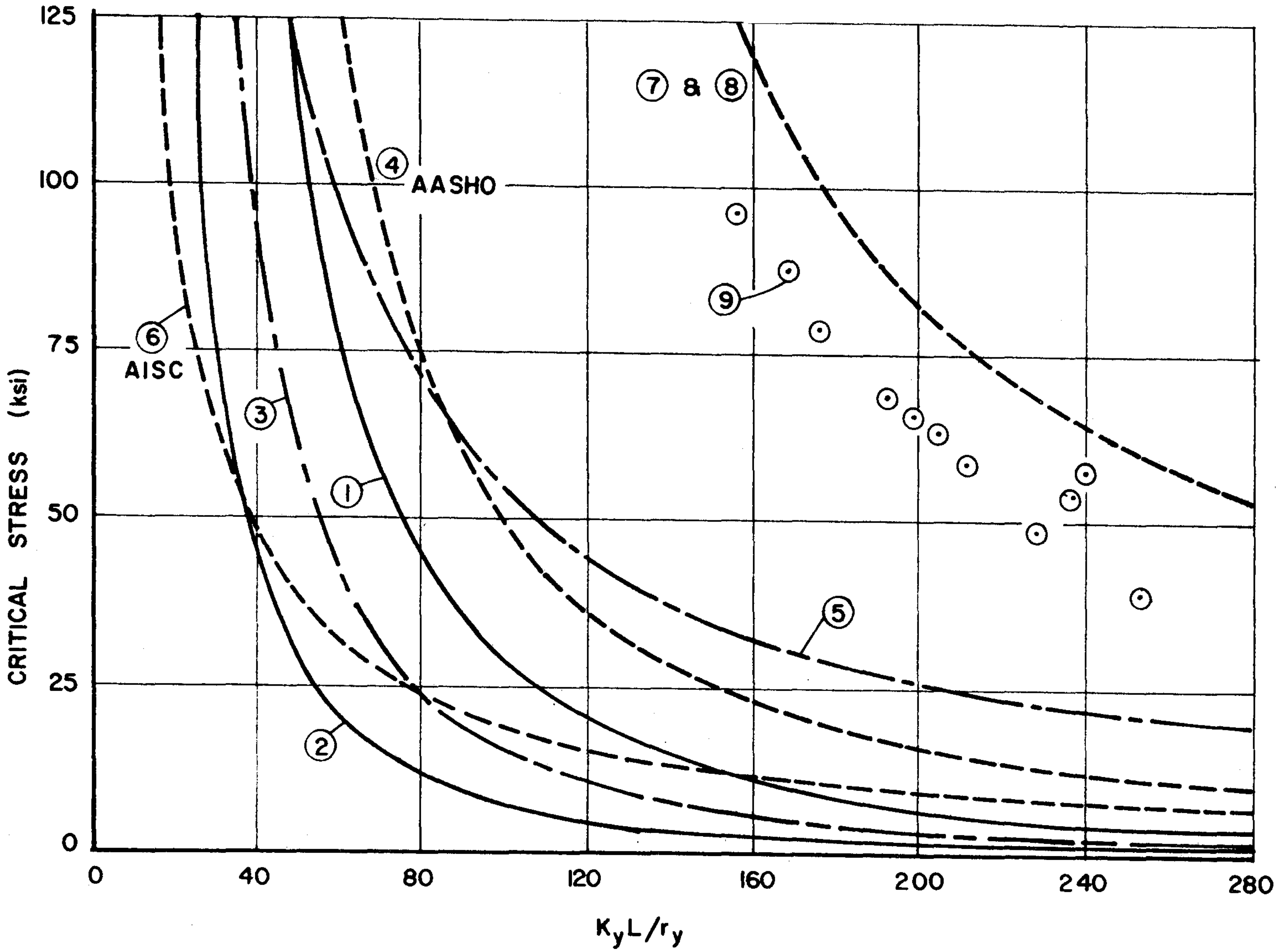

FIGURE 9. COMPARISON OF CRITICAL STRESSES vs. EFFECTIVE L/r_y FROM VARIOUS THEORTICAL FORMULAS AND TEST RESULTS

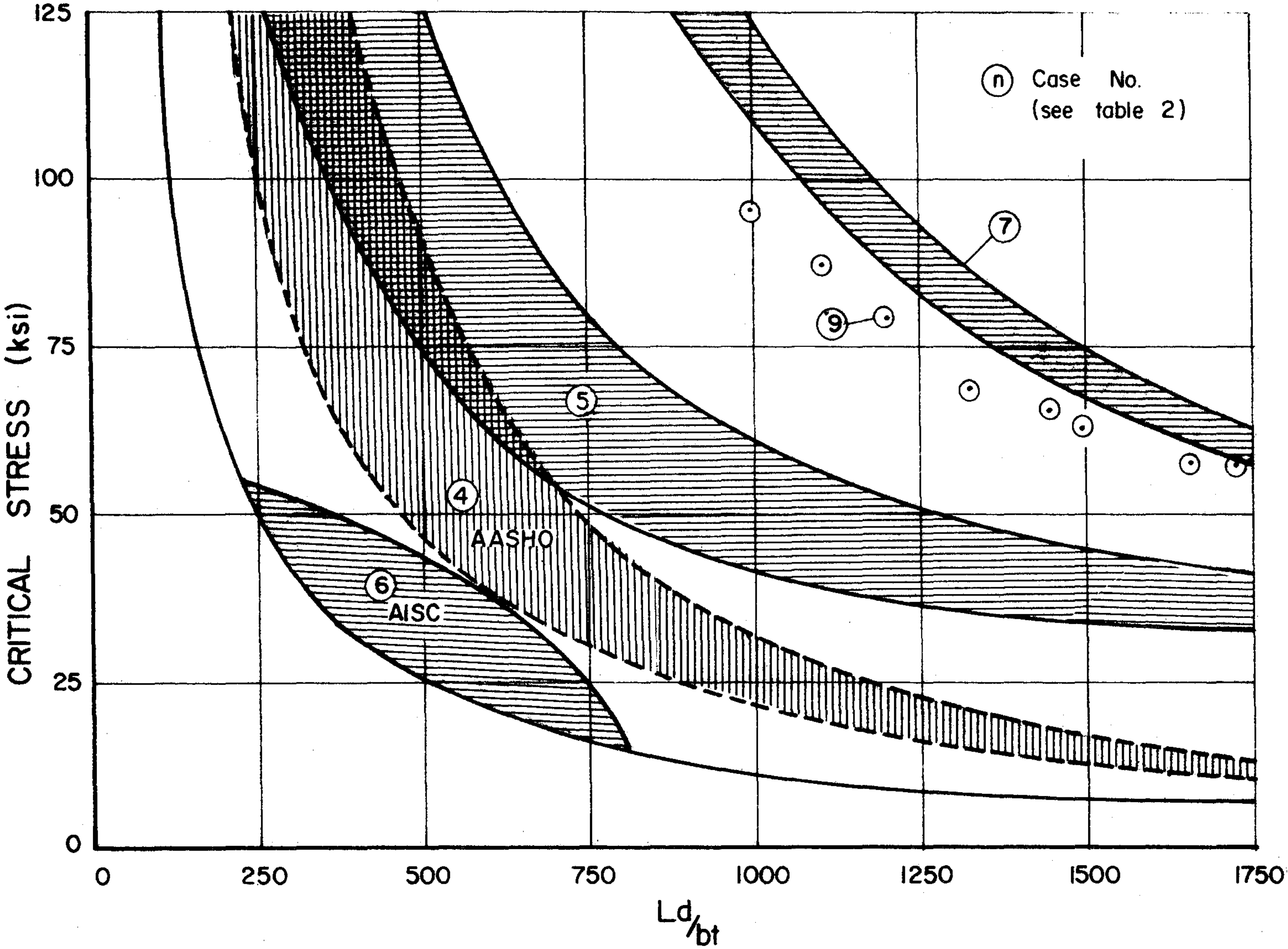

FIGURE 10. RANGES OF CRITICAL STRESSES vs. Ld/bt FOR VARIOUS THEORETICAL FORMULAS.

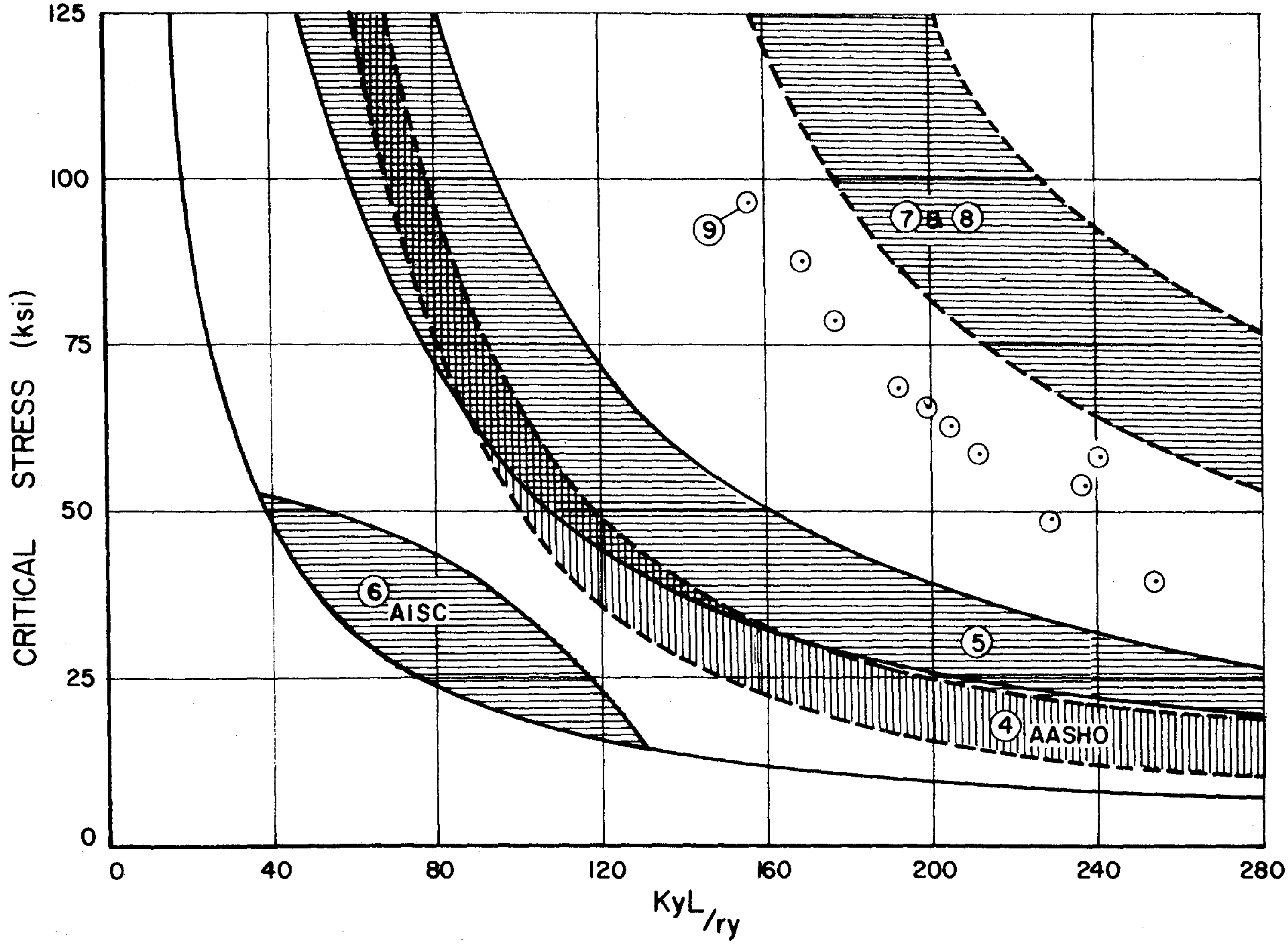

FIGURE 11. RANGE OF CRITICAL STRESSES vs. KyL/ry FOR VARIOUS THEORETICAL FORMULAS.

sampling of standard I and WF shapes.

Cases 4, 5, 6, 7, and 8 refer to lateral-torsional buckling cases and show a relationship to both Ld/bt and K_yL/r_y ratios. These cases fall within a band on the Ld/bt and K_yL/r_y plots (See Figures 10 and 11). Cases 1, 2, and 3 refer to axial buckling and thus show a clear relationship to K_yL/r_y and fall within a band on the Ld/bt plots.

As is readily apparent, the conditions of loading and the end conditions have a significant effect on the critical stresses. It is interesting to note on Figures 8, 9, 10, and 11 that Equation (37) follows the Krefeld et al.,[4] results very well, as indeed it should, since Equation (37) was derived to account for end conditions and loading similar to those in reference 4. If a factor of safety is applied to Case 4 (AASHO suggests a safety factor of 1.8), this curve would be even further away from the test results.

TAPERED COLUMNS

When the load on a cantilever beam is limited by yielding at the support, it is apparent that some savings in material (and thus a savings in weight) can be made by tapering the web depth and flange width. A series of load tests were conducted on cantilever beams with partial end restraint by Krefeld et al.[4] The beam loadings were end point loads and quarter-point loads. A lateral end brace was attached to the tension flange which allowed rotation of the end, but did not permit the tension flange to move laterally (See Figure 12). This lateral brace restricted the motion of the compression flange also. This test set-up more closely approximates the loading and end conditions for an overhead sign bridge than those in any of the theoretical

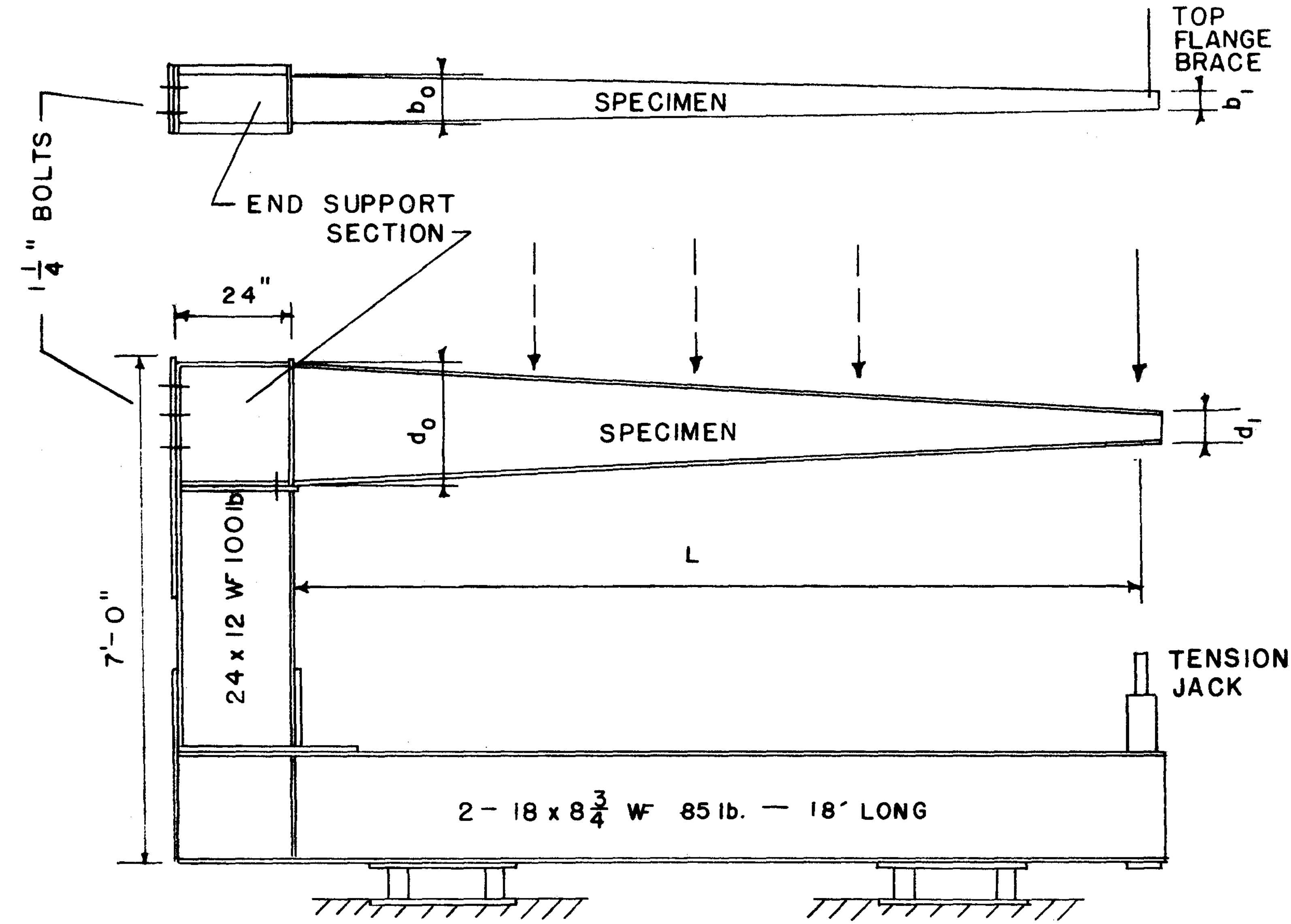

FIGURE 12. KREFELD *et. al.*[4] TESTING ASSEMBLY

derivations. And, as is shown on Figures 8 and 9, the failure stresses are much larger than any of the previously derived theoretically critical stresses.

The test results yielded the following empirical equations:

for end load and end brace:

$$f_{cr} = \frac{80,000,000}{\frac{Ld}{bt}} \text{ psi} \tag{40}$$

for $5000 > \frac{Ld}{bt} > 2500$

and

$$f_{cr} = \frac{110,000,000}{\frac{Ld}{bt}} - 7000 \text{ psi} \tag{41}$$

for $5000 > \frac{Ld}{bt} > 1000$

where f_{cr} is the nominal stress at the support when elastic buckling occurs.

For equally spaced four-point loads, simulating uniformly distributed loading, with end brace.

$$f_{cr} = \frac{130,000,000}{\frac{Ld}{bt}} \text{ psi} \tag{42}$$

for $5000 > \frac{Ld}{bt} > 3500$

$$f_{cr} = \frac{216,000,000}{\frac{Ld}{bt}} - 20,000 \text{ psi} \tag{43}$$

for $5000 > \frac{Ld}{bt} > 2000$

Equations (40) and (42), although simpler, become increasingly conservative for high yield point materials below the limits of Ld/bt specified.

It is possible with tapered beams to have stresses at points along the beam which are greater than the stress at the support. This is illustrated in Figure 13. Thus, it becomes necessary to apply a reduction factor, R,

$$R = f\left[\left(\frac{Z_o}{Z_1}\right), \left(\frac{b_o\, d_1}{b_1\, d_o}\right)\right] \tag{44}$$

where Z_o/Z_1 is the ratio of section moduli at the fixed and free ends.

The reduction factor, R, is defined as the ratio of the nominal stress at the support of a tapered beam to that of an untapered beam having the same section at the support, when elastic buckling occurred.

The critical stress producing buckling of a tapered beam with a given span was found to vary with the parameter,

$$\alpha = \frac{Z_o}{Z_1}\left(\frac{b_o\, d_1}{b_1\, d_o}\right)^{3/2} \tag{45}$$

The reduction factor can be expressed in terms of α as follows;

for end load, with end brace:

$$R = \frac{7 + \alpha}{5 + 3\alpha} \tag{46}$$

for equally spaced four-point loading with end brace:

$$R = \frac{4 + \alpha}{3 + 2\alpha} \tag{47}$$

From these formulas, it is possible to determine maximum tapers

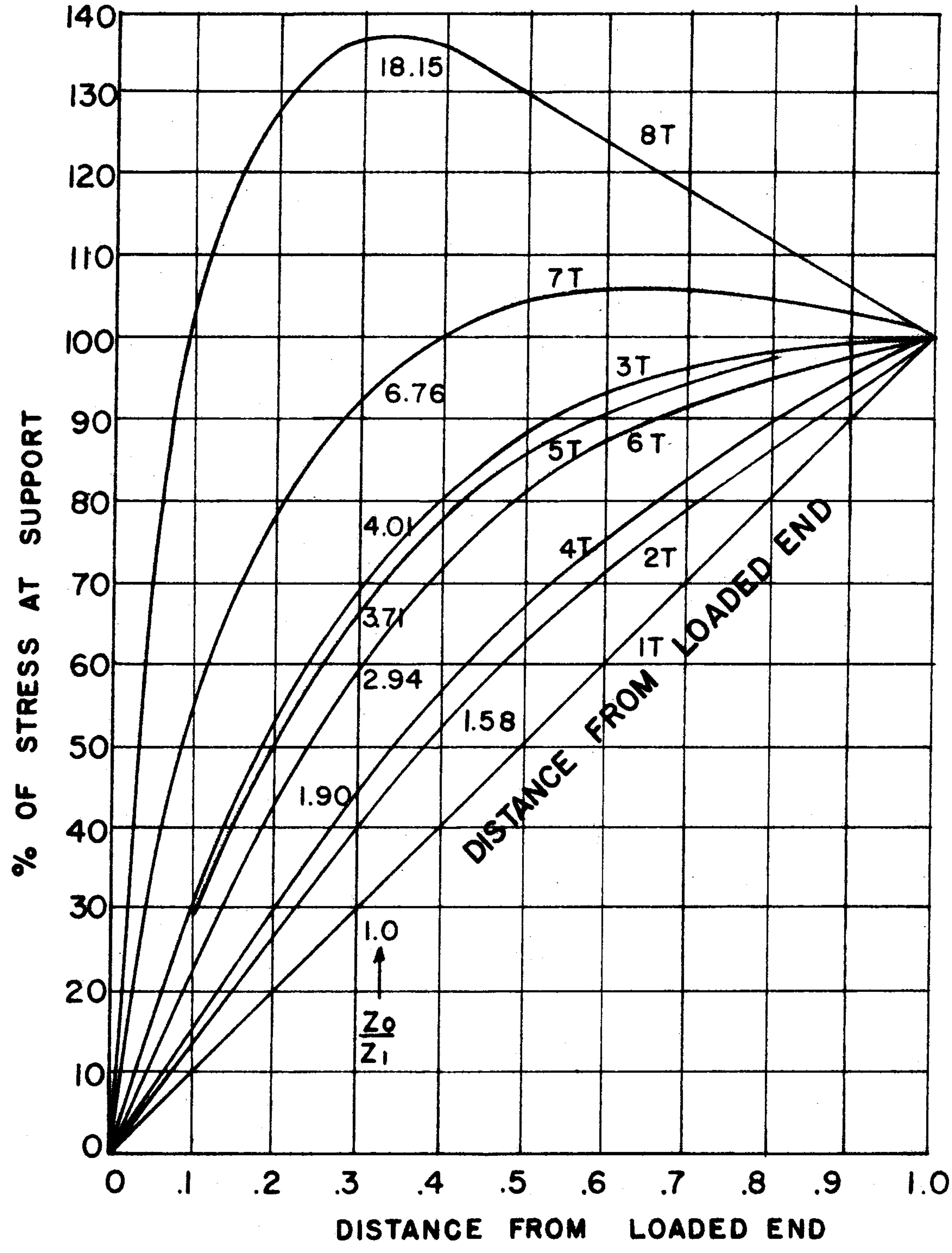

FIGURE 13. FLANGE STRESS DISTRIBUTION—BOTH FLANGES AND WEB TAPERED—SINGLE END LOAD (FROM KREFELD *et. al.*[4])

such that no reduction is necessary. Note that when $R = 1.0$, $\alpha = 1.0$. Therefore, no reduction is necessary when

$$\alpha = 1.0 = \frac{Z_o}{Z_1} \left(\frac{b_o \, d_1}{b_1 \, d_o} \right)^{3/2} \tag{48}$$

If there is an axial load in addition to the lateral load, then the taper must be less than that calculated by the above formula. A reasonable upper limit appears to be $Z_o/Z_1 = 4.0$.

PROPOSED CRITICAL STRESS FORMULA

The critical stress formulas (Equation 30 and Equation 37) can now be rewritten to include the effects of taper on the column. The effects of eccentricities of loading, residual stresses, and imperfections are considered to be adequately compensated for in the factor of safety.

The proposed critical stress formula to be used for overhead sign bridge supports is

$$f_{cr} = \frac{R}{I_x} \left(\frac{2d}{L} \right) \left(\frac{K_x}{K_y} \right)^2 \frac{\sqrt{EI_y GK}}{\left(1 - \frac{d}{2L} \sqrt{\frac{EI_y}{GK}} \right)} \tag{49}$$

where R is the reduction factor for taper obtained from Equation (46) or Equation (47), K_y is determined using Equation (35) and Figure C1.8.3 of the *AISC Manual of Steel Construction*[3], and the torsional rigidity, K, is

$$K = \frac{1}{3} \, d t_w^{\,3} + \frac{2}{3} \, b t_f^{\,3} \tag{50}$$

Thus, the maximum allowable stress to use in the interaction formulas (Equations 52, 53, 54 and 58) will be

$$F_b, F_{bC} = \frac{1}{F.S.} \quad \frac{R}{I_x} \left(\frac{2d}{L}\right)\left(\frac{K_x}{K_y}\right)^2 \frac{\sqrt{EI_y GK}}{\left(1 - \frac{d}{2L}\sqrt{\frac{EI_y}{GK}}\right)} \tag{51}$$

THE INTERACTION FORMULAS

The interaction formulas in the *AISC Steel Construction Manual*[3] (Formulas 6, 7a, and 7b) are based on F_b values calculated from AISC[3] Formulas 4 and 5. Formula 5 is based on lateral buckling of a simple beam under pure bending and, as such, does not have a provision for effective length. Formula 4 is based on the buckling strength of the compression flange plus one-sixth of the web.

AISC (6) $$\frac{f_a}{F_a} + \frac{f_b}{F_b} \leq 1.0 \quad (f_a/F_a \leq 0.15) \tag{52}$$

AISC (7a) $$\frac{f_a}{F_a} + \frac{C_m f_b}{\left(1 - \frac{f_a}{F_e'}\right) F_b} \leq 1.0 \quad (f_a/F_a > 0.15) \tag{53}$$

AISC (7b) $$\frac{f_a}{0.6F_y} + \frac{f_b}{F_b} \leq 1.0 \quad \text{(at brace points)} \tag{54}$$

AISC (4) $$F_b = \left[1.0 - \frac{\left(\frac{L}{r_{yf}}\right)^2 F_y}{4\Pi^2 EC_b}\right] 0.60\ F_y \tag{55}$$

$$\text{AISC (5)} \qquad F_b = \frac{12{,}000{,}000}{\frac{Ld}{bt}} \tag{56}$$

where L is the unbraced length of the compression flange, r_{yf} is given by Equation (10). C_b, which can conservatively be taken as unity, is equal to

$$C_b = 1.75 - 1.05\left(\frac{M_1}{M_2}\right) + 0.3\left(\frac{M_1}{M_2}\right)^2 \leq 2.3 \tag{57}$$

where M_1 is the smaller and M_2 is the larger of the moments at the ends of the unbraced length, taken about the strong axis of the member.

It is our suggestion, at this time, that the present form of the interaction equations be retained, but that the F_b values be chosen as the larger of F_b from Formulas 4, 5, or Equation (51).

An alternate suggestion is to separate the bending stresses caused by the forces into two parts; one due to cantilever type action, and the other due to simple beam bending action. The interaction formula would then take the form:

$$\frac{f_a}{F_a} + \frac{f_{bC}}{F_{bC}} + \frac{f_{bS}}{F_{bS}} \leq 1.0 \tag{58}$$

where F_{bC} would be obtained from Equation (51) and F_{bS} from AISC[3] Formulas 4 or 5. Note that F_a is a function of K_yL/r_y, F_{bC} is a function of K_yL/r_y and K_xL/r_x, and F_{bS} is a function of Ld/bt or L/r_{yf}. This form is useful from the standpoint that a cantilever action may occur about the strong axis at the same time as a simple beam bending action occurs about the weak axis in the case of an overhead sign bridge support.

THE NONLINEAR COMPUTER PROGRAM

The determination of the critical buckling load (usually called the Euler Buckling Load) can be handled as a linear problem in mechanics. This type of solution yields one specific load at which the structure should theoretically fail. However, this method of determining the buckling load uses many assumptions which may not be applicable for most practical structures. Thus, the loads obtained from these mathematically exact (but sometimes practically impossible) formulations yield critical loads that are larger than experimental results (sometimes many times larger). The engineer knows that these results must be modified by appropriate safety factors before they can be used with confidence. In many cases, the safety factors used are unnecessarily large because of the many uncertainties of loading, geometry and initial stresses which must be allowed for.

To more accurately obtain maximum loads and stresses for practical structures, it is necessary to use a more accurate formulation of the problem. This formulation includes nonlinear terms. The problem becomes more complex now because the load-displacement relationship will no longer be linear. It also means that each structure and each set of loads becomes a unique problem and great care must be used to correlate results of one problem with another. The maximum loads and stresses now are not determined from a single load. Rather, the load-displacement curve is plotted and the maximum load is ascertained by the load at which the deflections begin to get very large (See

Figure 14).

The determination of a single critical load in nonlinear problems can thus occur only if there is only one load on the structure. Usually the loads are applied in loading sets. In this case there does not exist a single critical load and the ratio of each load to the others is the only basis for comparison which one has. Thus, in order to develop any type of design criteria, it seems logical to attempt to relate a large number of nonlinear solutions to one another by a consideration of critical stresses rather than critical loads. The complicating factor in this approach is that the stresses calculated from a nonlinear solution may not show a clear relationship to those calculated by the linear theory. However, it is proposed that a critical stress formula of the form

$$f_{cr} = \frac{C_1 \left(\frac{2d}{L}\right)\left(\frac{K_x}{K_y}\right)^2 \sqrt{EI_y GK}}{I_x \left(1 - \frac{d}{2L} \sqrt{\frac{EI_y}{GK}}\right)} \qquad (59)$$

be used for design purposes. The term C_1 involves effects of residual stresses, a safety factor, eccentricities, imperfections, and effects of column taper.

The present status of the nonlinear deflection analysis is this: (1) the elastic and axial nonlinear matrices have been checked and the torsional nonlinear matrix has been derived; (2) the computer program has been written and debugged; and (3) a few cases have been run to test the program: (a) axial load buckling (small initial eccentricity

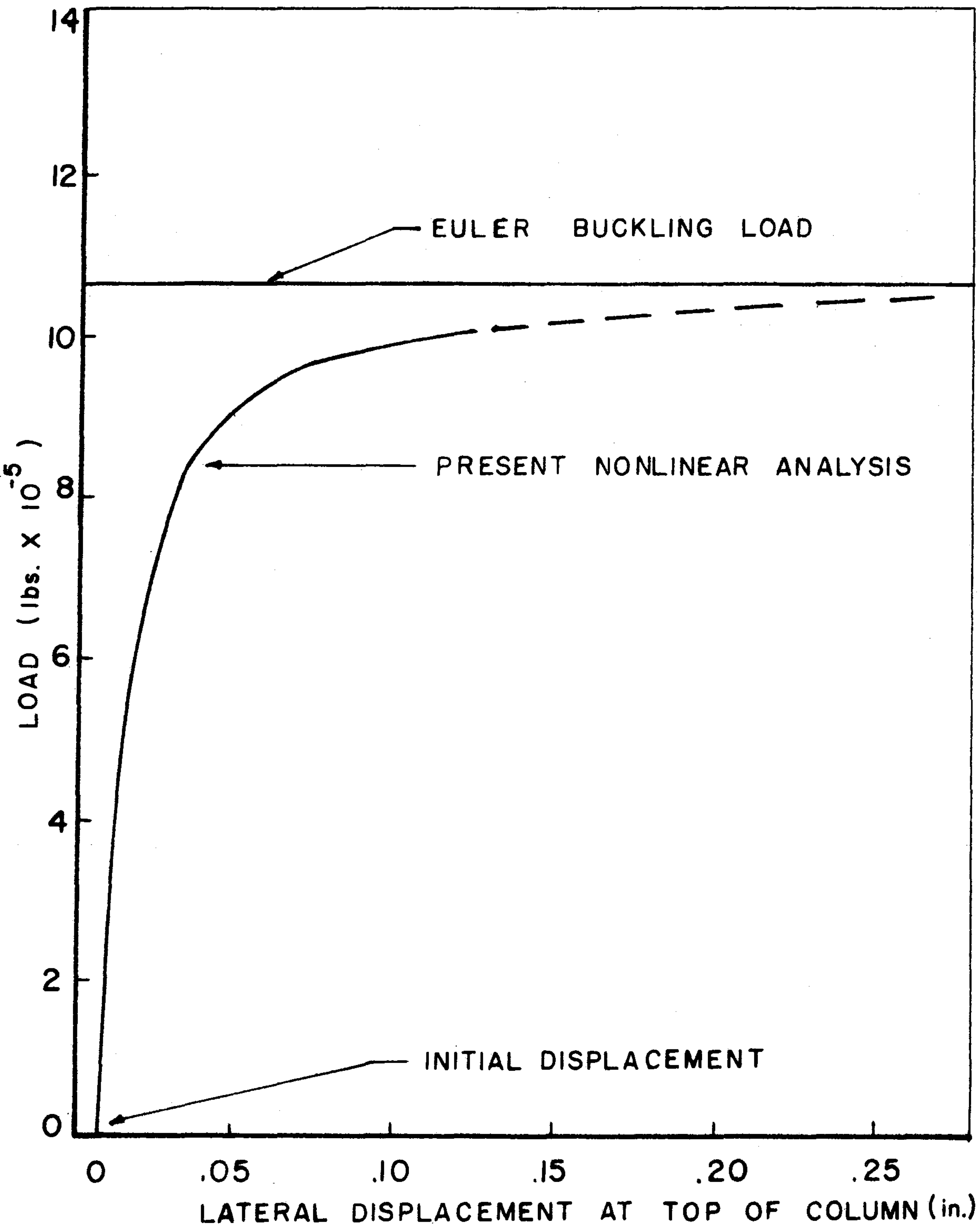

FIGURE 14. COMPARISON OF NONLINEAR ANALYSIS AND THEORETICAL BUCKLING LOAD FOR AXIALLY LOAD COLUMNS.

to trigger buckling) (See Figure 14), (b) axial load with heavy lateral load, (c) lateral load only, and (d) lateral load only and a partial end restraint under the load (to compare with the test results by Krefeld _et al_.[4]).

The program calculates both deflections and stresses. A plot of load versus deflections will indicate the point of impending buckling and the stress computations indicate whether the yield strength has been exceeded or not.

The solution for critical stresses in beam-columns under general loading conditions has not been accomplished to date. These problems exist frequently in practice, but simplifying assumptions are resorted to in order to avoid solving such a complex problem.

At the outset, the authors believed that the determination of critical stress for a problem illustrated by Figure 15 did not lend itself to solution by an eigenvalue approach. Thus, a nonlinear, large deflection analysis was undertaken. The results from such an analysis (See Figures 14 and 15) indicate buckling as the point at which the deflections become large. A nonlinear, large deflection analysis involves many problems not encountered in the solution of a linear, small deflection structures problem. Among the added difficulties are iteration limits, convergence criteria, effects of small lateral loads, number of load steps, etc.

The nonlinear analysis was based on a finite element approach. The stiffness matrices for the beam sections had to be derived because the standard matrices do not include warping of the cross section. This is a very important quantity if one is attempting to

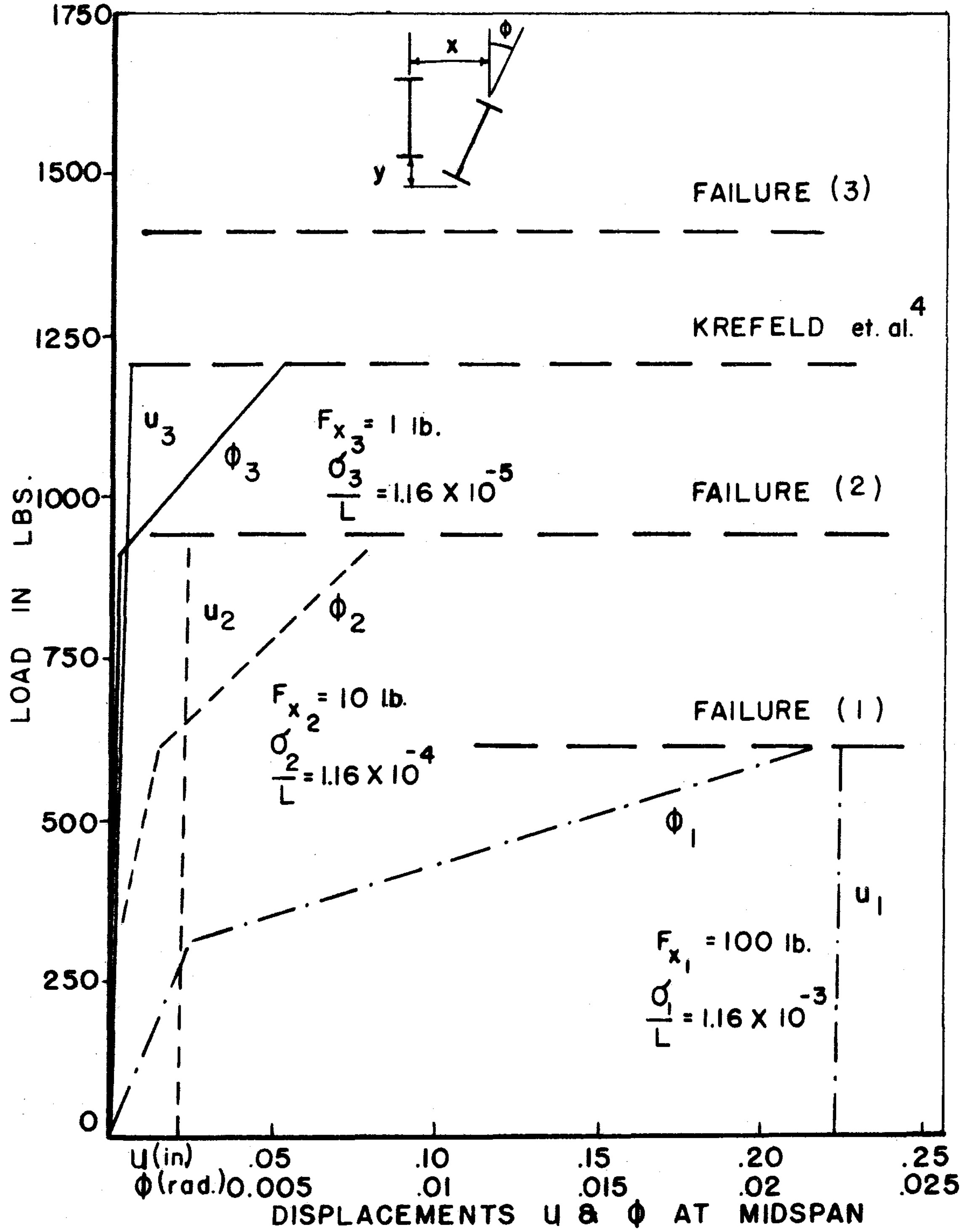

FIGURE 15. NONLINEAR LOAD-DISPLACEMENT CURVES FOR 12 JR 11.8 CANTILEVER BEAM WITH END RESTRAINT (Krefeld, et. al.[4] Test conditions)

determine critical lateral buckling loads.

CRITERIA FOR TRIGGER LOADS

In this nonlinear program, a certain amount of initial displacements must exist to cause the lateral displacements to become large as the applied load reaches the buckling load. However, the choice of the value of the initial out-of-straightness is a delicate matter. If the value is too high, the member will fail at a load somewhat lower than the theoretical value. If the value is too low, the large displacements resulting from a build up of the nonlinear effects will not occur at all and the member will simply continue to deform linearly.

The initial displacements can be inserted into the program in two ways: (1) as initial displacements, or (2) as loadings which cause equivalent initial displacements. The present program uses the second approach and these loads which, in effect, initiate the buckling are called the "trigger loads". After a number of attempts, it has become apparent that a trigger load which causes a δ/L (or a rotation about the weak axis) on the order of 10^{-5}, is sufficient. A number smaller than this is lost in the round-off unless double precision is used in the computer and a larger value results in a failure load below the theoretical and/or actual load (See Figure 15).

The nonlinear study indicates the drastic reduction in allowable loads in the presence of initial displacements or small weak axis direction loads. This is to be expected based on the secant formula expression which is shown in Figure 16.

Most nonlinear analyses require a thorough knowledge of the

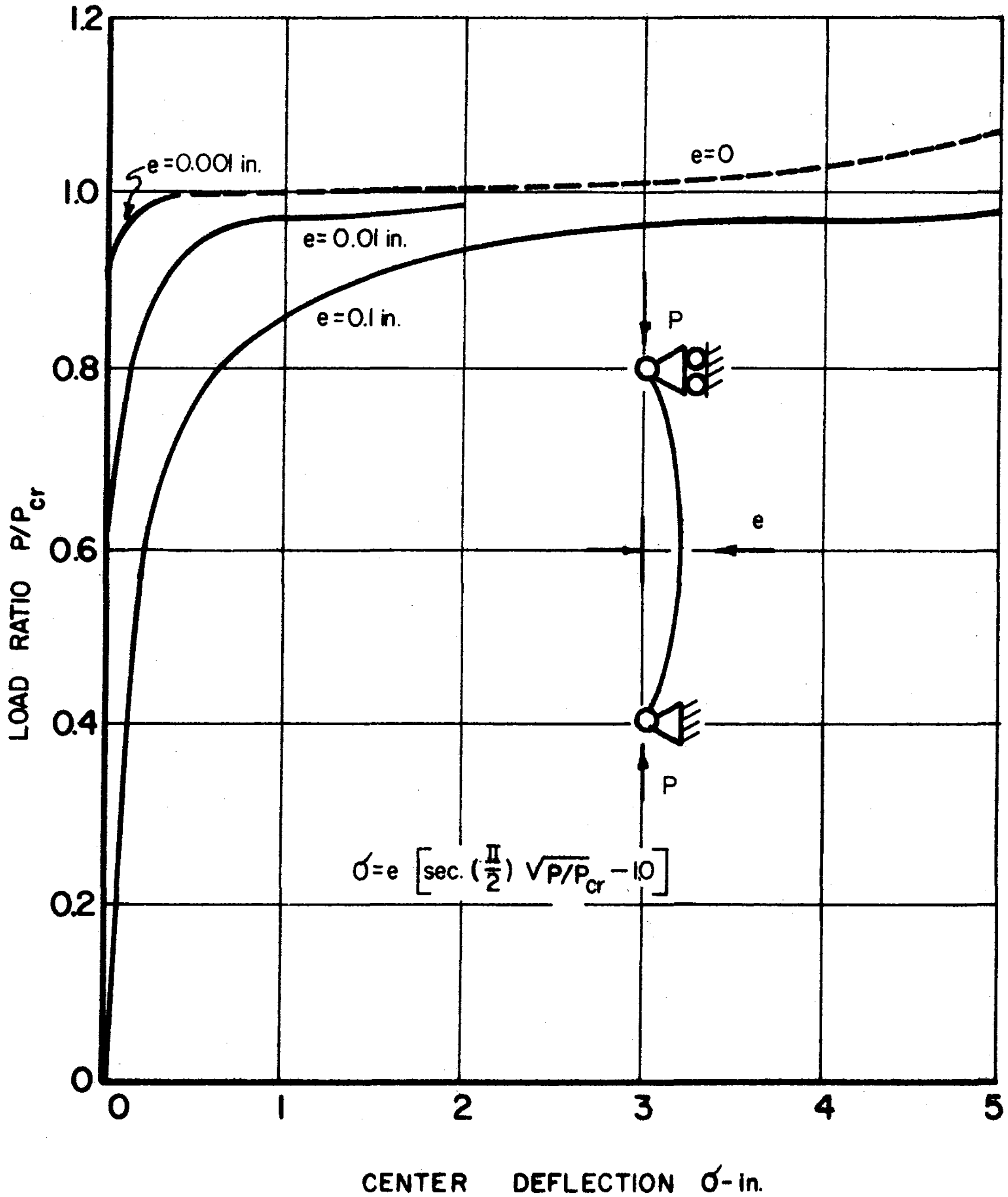

FIGURE 16. EFFECTS OF INITIAL DEFLECTIONS — SECANT FORMULA

problem to be solved. In fact, the solution usually must be known (within bounds) before a solution can be obtained in a reasonable amount of time. These programs are expensive to run.

The results from this initial study indicate that there may exist an eigenvalue solution for the lateral buckling problem and possibly for the general loading case. If this could be derived, a large savings in computer time could be effected in the solution for critical loads. The authors are presently working on this problem, but results are not available at the present time. The development of an eigenvalue routine for determining critical loads for a beam-column under general loading conditions would make an excellent topic for future study if this present study is to be carried further.

There may be many cases in which a loading combination exists in which each load does not develop appreciable stresses in the members, but the combination causes buckling. These combinations are difficult to predict. One way to determine these "worst case loadings" is to compute a large number of critical loads in various combinations and compare the results. A study of this magnitude is outside the realm of possibility for the present study.

A more detailed discussion of the finite element, nonlinear program is given in the Appendix.

CONCLUSIONS AND RECOMMENDATIONS

This report considers the problem of critical stresses in open sections. The major conclusion being that the present AASHO critical stress formula is overly conservative when applied to sign supports.

A critical stress formula was developed which is applicable to restrained cantilevers which is a close approximation to the boundary conditions existing on an overhead sign bridge. Thus, a substitute critical stress formula has been proposed.

There are areas which should be explored in more detail before the recommendations discussed herein could be accepted into the AASHO code requirements. These include: (a) full-scale tests on large wide flange shapes to determine critical stresses under various loadings and end restraints: (an extension of the Krefeld _et al._[4] study) (b) development of a finite element buckling program to include effects of lateral-torsional buckling, (c) correlation of proposed critical stress formula (Equation 49) with (a) and (b) above, and (d) determine the validity of the use of an interaction formula of the form of Equation (58).

It is apparent that application of the AASHO code requirements results in a grossly overdesigned double-tapered column as tested by Krefeld _et al._, and as shown in Figures 8 and 9.

The buckling problem is obviously very dependent upon the conditions of end restraint. Thus, code requirements for critical stress should provide means for accounting for varying end conditions. The proposed formula has this flexibility. Limiting stresses based on worst case conditions (such as the lateral buckling of a beam under pure bending) can "build in" a real factor of safety many times larger than intended. This is particularly true when the conditions at the supports are different. This is obvious from the critical axial stress formula

$$f_{cr} = \frac{\Pi^2 E}{\left(\frac{KL}{r}\right)^2} = C \frac{\Pi^2 E}{\left(\frac{L}{r}\right)^2} \tag{49}$$

which leads to C values of 4 and 1 for fixed end, and pinned end cases of end restraint, respectively. Thus, a factor of safety of four would be "built in" to the fixed end case if the pinned end formula was applied to a problem with fixed ends. This is analogous to the problem discussed in this report.

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REFERENCES

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NOMENCLATURE

NOMENCLATURE

A, A_f, A_w	Areas of the cross section, flange and web, respectively.
b	Width of flange.
c	Distance from neutral axis to edge of beam (usually d/2).
d	Depth of beam.
E	Modulus of elasticity.
F_a, F_b	Maximum allowable stresses in axial and bending, respectively.
F_y	Yield point stress.
F.S.	Factor of safety.
f	Calculated or actual stresses.
G	Torsional modulus.
G_A, G_B	End condition parameters used in determining effective lengths (See Equation 35).
I_x, I_y	Moments of inertia about the x and y axes, respectively.
K	Torsional rigidity (not equal to the polar moment of inertia for open sections, See Equation 50).
K_x, K_y, K_z	Effective length factors for bending about the x, y, and z axes, respectively.
$\frac{K_y L}{r_y}$, $\frac{L}{r_{yf}}$, $\frac{Ld}{bt}$	Buckling parameters appearing in the critical stress formulas.
L	Unsupported length.
M_o	Applied end moment.
P	Applied point load.
R	Stress reduction factor for tapered beams.

r_x, r_y, r_z	Radii of gyration about the x, y, and z axes, respectively. ($r = \sqrt{I/A}$).
r_{yf}	Radius of gyration of one flange plus one-sixth the area of the web (See Equation 10).
t_f, t_w	Thickness of flange and web, respectively.
u, v, w	Displacements in the x, y, and z directions, respectively.
x, y, z	Coordinate axes.
α	Taper parameter.
γ	Critical load parameter.
δ	Displacement.
ψ, θ, ϕ	Rotations about the x, y, and z axes, respectively.

Subscripts

a	Refers to axial stress.
b	Refers to bending stress.
bC, bS	Refers to bending stresses due to cantilever action and simple beam action, respectively.
cr	Refers to critical stress.
f, w	Denotes the flange and web, respectively.
x, y, z	Refers to the x, y, and z axes, respectively.
0, 1	Denotes support point and end point, respectively, on a tapered beam.
$\doteq$, $\cong$	Approximately equal to.

APPENDIX

THE NONLINEAR, LARGE DEFLECTION PROGRAM

The solution of critical stresses in beam-columns under general loading conditions has not been accomplished to date. Although these problems often exist in practice, a solution has not been formulated because of the complexity of the problem. Simple, special cases have been solved and these are discussed in the body of the report.

A finite element analysis was undertaken to determine the critical stresses. This required the development of a new stiffness matrix which allowed for warping of the cross section. The beam stiffness matrices which are available in current literature do not permit warping of the cross section. For the purposes of structural analysis, warping is not a factor, but it is a very important quantity to consider in the lateral buckling problem.

The independent variables assumed in the analysis are (See Figure A-1):

u - displacement in x-direction

θ - rotation about y-axis

v - displacement in y-direction

ψ - rotation about x-axis

w - displacement in z-direction

ϕ - rotation about z-axis

α - change in rotation about z-axis.

These quantities yield expressions which are cubic in bending about the x and y axes, linear in axial deformation and cubic in torsional deformation.

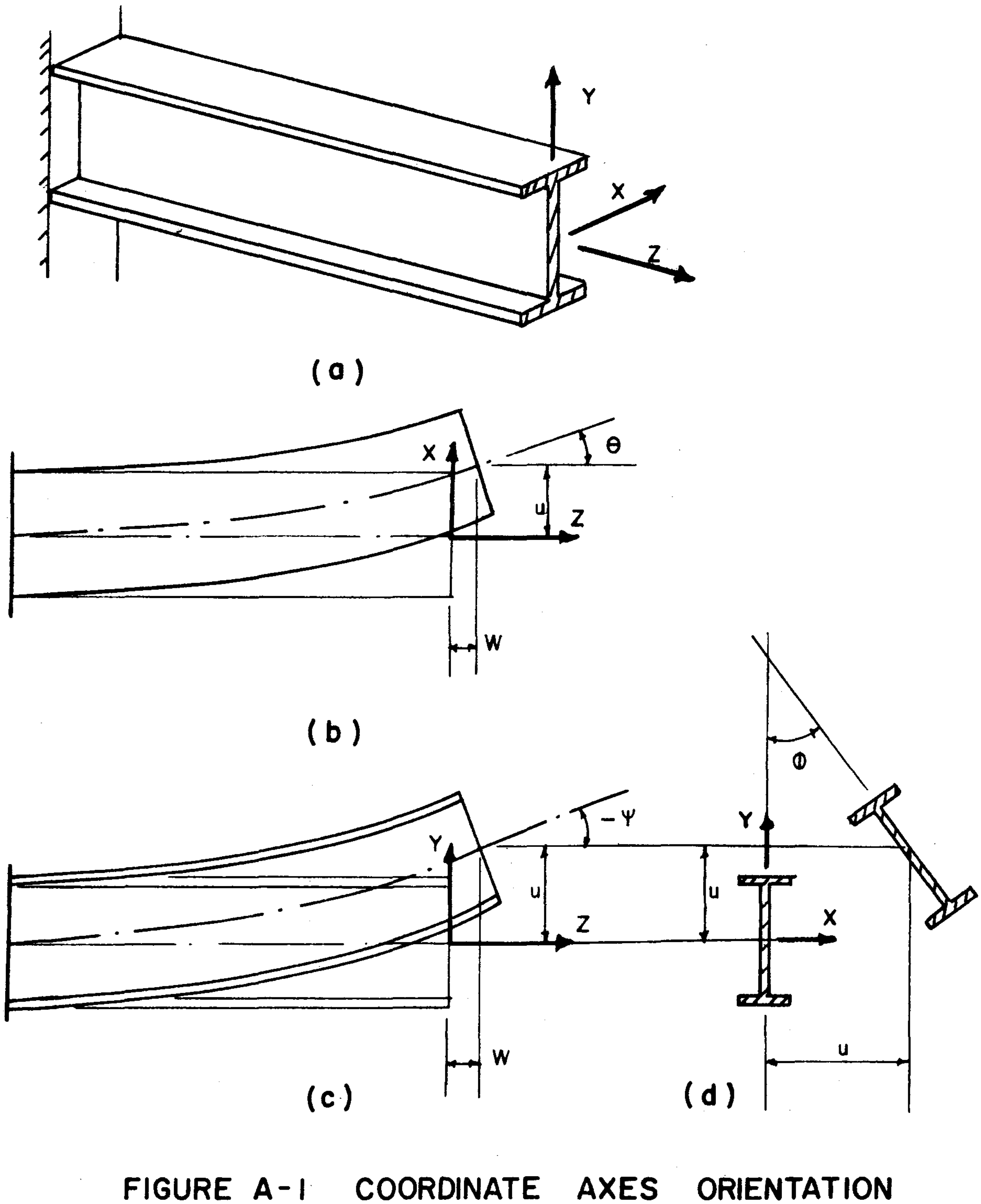

FIGURE A-1 COORDINATE AXES ORIENTATION FOR A COLUMN AND DEFORMED SHAPES IN X-Z , Y-Z AND X-Y PLANES.

The nonlinear strain-displacement relationships are of the following form.

$$\varepsilon_{zz} = w,_z + \frac{1}{2} \{(u,_z)^2 + (v,_z)^2\} \quad \text{(A1)}$$

$$\varepsilon_{xz} = u,_z + w,_x + (u,_x)\ (u,_z) + (v,_x)\ (v,_z) + (w,_x)\ (w,_z) \quad \text{(A2)}$$

$$\varepsilon_{yz} = v,_z + w,_y + (u,_y)\ (u,_y) + (v,_y)\ (v,_z) + (w,_y)\ (w,_z) \quad \text{(A3)}$$

where the comma refers to partial differentiation. The displacement functions were substituted into the above strain displacement functions. It was assumed that ε_{xz} is negligible in the web and that ε_{yz} is negligible in the flanges.

The strain-displacement terms were then substituted into the strain energy expression.

$$U = \frac{1}{2} \int_{vol.} \{E\ (\varepsilon_{zz})^2 + G\ (\varepsilon_{xz})^2 + G\ (\varepsilon_{yz})^2\}\ dAdz \quad \text{(A4)}$$

This expression was calculated separately for the web and the flanges.

The method of virtual work was applied to the total energy expression to evaluate the load-displacement relationships. This results in

$$[K_E]\ \{q\} + \{Q_A\} + \{Q_T\} = \{Q\} \quad \text{(A5)}$$

where

$[K_E]$ - is the elastic stiffness matrix

$\{q\}$ - is the displacement vector

$\{Q_A\}$ - are the forces due to nonlinear displacements associated with axial deformations

$\{Q_T\}$ - are the forces due to nonlinear displacements associated with torsional deformations

$\{Q\}$ - is the applied load vector

The individual terms for a single element are shown in Equations A6, A7, and A8.

where

$$C_T = E \{\frac{d^2}{4} I_y + \frac{1}{144} (A_w{}^3 + 2A_f{}^2)\}$$

A_w = area of web

A_f = area of flange

$$C_A = E \{I_y + I_{xw} + \frac{1}{2} K_f + \frac{K}{2(1 + \mu)} + \frac{d^2A_f}{2} \}$$

$$C_x = \frac{1}{2} EI_{xw} + \frac{1}{4} EK_f - GK_f$$

$$C_y = - \frac{1}{2} EI_y + \frac{1}{2} GK_w$$

$$K_w = \frac{1}{3} dt_w{}^3$$

$$K_f = \frac{1}{3} bt_f{}^3$$

$$
[K_E]\{q\} =
\begin{bmatrix}
\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & & & & & & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & & & & & \\
\frac{6EI_y}{L^2} & \frac{4EI_y}{L} & & & & & & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} & & & & & \\
 & & \frac{12EI_x}{L^3} & -\frac{6EI_x}{L^2} & & & & & & -\frac{12EI_x}{L^3} & -\frac{6EI_x}{L^2} & & & \\
 & & -\frac{6EI_x}{L^2} & \frac{4EI_x}{L} & & & & & & \frac{6EI_x}{L^2} & \frac{2EI_x}{L} & & & \\
 & & & & \frac{12C_T}{L^3}+\frac{6GK}{5L} & \frac{6C_T}{L^2}+\frac{GK}{10} & & & & & & -\frac{12C_T}{L^3}-\frac{6GK}{5L} & \frac{6C_T}{L^2}+\frac{GK}{10} & \\
 & & & & \frac{6C_T}{L^2}+\frac{GK}{10} & \frac{4C_T}{L}+\frac{2GKL}{15} & & & & & & -\frac{6C_T}{L^2}-\frac{GK}{10} & \frac{2C_T}{L}-\frac{GKL}{30} & \\
 & & & & & & \frac{EA}{L} & & & & & & & -\frac{EA}{L} \\
-\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & & & & & & \frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & & & & & \\
\frac{6EI_y}{L^2} & \frac{2EI_y}{L} & & & & & & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} & & & & & \\
 & & -\frac{12EI_x}{L^3} & \frac{6EI_x}{L^2} & & & & & & \frac{12EI_x}{L^3} & \frac{6EI_x}{L^2} & & & \\
 & & -\frac{6EI_x}{L^2} & \frac{2EI_x}{L} & & & & & & \frac{6EI_x}{L^2} & \frac{4EI_x}{L} & & & \\
 & & & & -\frac{12C_T}{L^3}-\frac{6GK}{5L} & -\frac{6C_T}{L^2}-\frac{GK}{10} & & & & & & \frac{12C_T}{L^3}+\frac{6GK}{5L} & -\frac{6C_T}{L^2}-\frac{GK}{10} & \\
 & & & & \frac{6C_T}{L^2}+\frac{GK}{10} & \frac{2C_T}{L}-\frac{GKL}{30} & & & & & & -\frac{6C_T}{L^2}-\frac{GK}{10} & \frac{4C_T}{L}+\frac{2GKL}{15} & \\
 & & & & & & -\frac{EA}{L} & & & & & & & \frac{EA}{L}
\end{bmatrix}
\begin{Bmatrix}
u_1 \\ \theta_1 \\ v_1 \\ \psi_1 \\ \phi_1 \\ \alpha_1 \\ w_1 \\ u_2 \\ \theta_2 \\ v_2 \\ \psi_2 \\ \phi_2 \\ \alpha_2 \\ w_2
\end{Bmatrix}
\quad (A6)
$$

$$\frac{\partial U_A}{\partial q} = \{Q_A\} = \left\{ \begin{array}{l}
\frac{EA\ (w_2 - w_1)}{L} \left[- \frac{6}{5L} (u_2 - u_1) + \frac{1}{10} (\theta_1 + \theta_2) \right] \\[2ex]
\frac{EA\ (w_2 - w_1)}{L} \left[- \frac{1}{10} (u_2 - u_1) + \frac{L}{30} (4\theta_1 - \theta_2) \right] \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[- \frac{6}{5L} (v_2 - v_1) - \frac{1}{10} (\psi_1 + \psi_2) \right] \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[\frac{1}{10} (v_2 - v_1) + \frac{L}{30} (4\psi_1 - \psi_2) \right] \\[2ex]
\frac{C_A}{L} (w_2 - w_1) \left[- \frac{6}{5L} (\phi_2 - \phi_1) + \frac{1}{10} (\alpha_1 + \alpha_2) \right] \\[2ex]
\frac{C_A}{L} (w_2 - w_1) \left[- \frac{1}{10} (\phi_2 - \phi_1) + \frac{L}{30} (4\alpha_1 - \alpha_2) \right] \\[2ex]
\qquad\qquad\qquad\qquad 0 \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[\frac{6}{5L} (u_2 - u_1) - \frac{1}{10} (\theta_1 + \theta_2) \right] \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[- \frac{1}{10} (u_2 - u_1) + \frac{L}{30} (4\theta_2 - \theta_1) \right] \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[\frac{6}{5L} (v_2 - v_1) + \frac{1}{10} (\psi_1 + \psi_2) \right] \\[2ex]
\frac{EA}{L} (w_2 - w_1) \left[\frac{1}{10} (v_2 - v_1) + \frac{L}{30} (4\psi_2 - \psi_1) \right] \\[2ex]
\frac{C_A}{L} (w_2 - w_1) \left[\frac{6}{5L} (\phi_2 - \phi_1) - \frac{1}{10} (\alpha_1 + \alpha_2) \right] \\[2ex]
\frac{C_A}{L} (w_2 - w_1) \left[- \frac{1}{10} (\phi_2 - \phi_1) + \frac{L}{30} (4\alpha_2 - \alpha_1) \right]
\end{array} \right\} \tag{A7}$$

$$0$$

where $C_A = E\ \{I_y + I_{xw} + \frac{1}{2}\ K_f + \frac{K}{2(1+\mu)} + \frac{d^2 A_f}{2}\ \}$

$$
\frac{\partial U_T}{\partial q} = \left\{ Q_T \right\} = \left\{ \begin{array}{l}
c_x\left\{\frac{1}{5L^2}\left[6(\phi_1-\phi_2)(\psi_1-\psi_2)+3(\alpha_1-\alpha_2)(v_1-v_2)\right]+\frac{1}{5L}\left[-\alpha_1(\psi_1+2\psi_2)+\alpha_2(\psi_2+2\psi_1)\right]\right\}+ \\
c_y\left\{\frac{3}{5L^2}\left[(v_1-v_2)(\alpha_1-\alpha_2)-(\phi_1-\phi_2)(\psi_1-\psi_2)\right]+\frac{2}{5L}\left[\psi_1\alpha_1-\psi_2\alpha_2\right]\right\} \\[2ex]
c_x\left\{\frac{3}{5L^2}(\phi_1-\phi_2)(v_1-v_2)-\frac{1}{5L}(\phi_1-\phi_2)(\psi_1+2\psi_2)-\frac{2}{5L}\alpha_1(v_1-v_2)+\frac{1}{3}\alpha_1\psi_1+\frac{1}{15}\alpha_1\psi_2-\frac{1}{30}\alpha_2(\psi_1-\psi_2)\right\}+ \\
c_y\left\{-\frac{6}{5L^2}(v_1-v_2)(\phi_1-\phi_2)+\frac{1}{5L}\left[(v_1-v_2)(\alpha_1-2\alpha_2)-(\phi_1-\phi_2)(\psi_1-2\psi_2)\right]+\frac{1}{3}\alpha_1\psi_1-\frac{1}{15}\alpha_2\psi_2-\frac{1}{30}(\psi_1\alpha_2+\psi_2\alpha_1)\right\} \\[2ex]
c_x\left\{\frac{3}{5L^2}\left[(u_1-u_2)(\alpha_1-\alpha_2)+(\phi_1-\phi_2)(\theta_1-\theta_2)\right]-\frac{2}{5L}(\theta_1\alpha_1-\theta_2\alpha_2)\right\}+ \\
c_y\left\{\frac{1}{5L^2}\left[-6(\phi_1-\phi_2)(\theta_1-\theta_2)+3(\alpha_1-\alpha_2)(u_1-u_2)\right]+\frac{1}{5L}\left[\alpha_1(\theta_1+2\theta_2)-\alpha_2(\theta_2+2\theta_1)\right]\right\} \\[2ex]
c_x\left\{\frac{6}{5L^2}(u_1-u_2)(\phi_1-\phi_2)-\frac{1}{5L}\left[(u_1-u_2)(\alpha_1-2\alpha_2)+(\phi_1-\phi_2)(\theta_1-2\theta_2)\right]+\frac{1}{3}\theta_1\alpha_1-\frac{1}{15}\theta_2\alpha_2-\frac{1}{30}(\theta_1\alpha_2+\theta_2\alpha_1)\right\}+ \\
c_y\left\{-\frac{3}{5L^2}(u_1-u_2)(\phi_1-\phi_2)-\frac{1}{5L}(\phi_1-\phi_2)(\theta_1+2\theta_2)+\frac{2}{5L}\alpha_1(u_1-u_2)+\frac{1}{3}\alpha_1\theta_1+\frac{1}{15}\alpha_1\theta_2-\frac{1}{30}\alpha_2(\theta_1-\theta_2)\right\} \\[2ex]
c_x\left\{\frac{6}{5L^2}(u_1-u_2)(\psi_1-\psi_2)+\frac{3}{5L^2}(\theta_1-\theta_2)(v_1-v_2)-\frac{1}{5L}\theta_1(\psi_1+2\psi_2)+\frac{1}{5L}\theta_2(\psi_2+2\psi_1)\right\}+ \\
c_y\left\{-\frac{6}{5L^2}(v_1-v_2)(\theta_1-\theta_2)-\frac{3}{5L^2}(\psi_1-\psi_2)(u_1-u_2)-\frac{1}{5L}\psi_1(\theta_1+2\theta_2)+\frac{1}{5L}\psi_2(\theta_2+2\theta_1)\right\} \\[2ex]
c_x\left\{\frac{3}{5L^2}(u_1-u_2)(v_1-v_2)-\frac{1}{5L}(u_1-u_2)(\psi_1+2\psi_2)-\frac{2}{5L}\theta_1(v_1-v_2)+\frac{1}{3}\theta_1\psi_1+\frac{1}{15}\theta_1\psi_2-\frac{1}{30}\theta_2(\psi_1-\psi_2)\right\}+ \\
c_y\left\{\frac{3}{5L^2}(v_1-v_2)(u_1-u_2)+\frac{1}{5L}(v_1-v_2)(\theta_1+2\theta_2)+\frac{2}{5L}\psi_1(u_1-u_2)+\frac{1}{3}\psi_1\theta_1+\frac{1}{15}\psi_1\theta_2-\frac{1}{30}\psi_2(\theta_1-\theta_2)\right\} \\[2ex]
\qquad\qquad\qquad\qquad\qquad\qquad 0 \\[2ex]
-c_x\left\{\frac{1}{5L^2}\left[6(\phi_1-\phi_2)(\psi_1-\psi_2)+3(\alpha_1-\alpha_2)(v_1-v_2)\right]+\frac{1}{5L}\left[-\alpha_1(\psi_1+2\psi_2)+\alpha_2(\psi_2+2\psi_1)\right]\right\}- \\
c_y\left\{\frac{3}{5L^2}\left[(v_1-v_2)(\alpha_1-\alpha_2)-(\phi_1-\phi_2)(\psi_1-\psi_2)\right]+\frac{2}{5L}\left[\psi_1\alpha_1-\psi_2\alpha_2\right]\right\} \\[2ex]
c_x\left\{-\frac{3}{5L^2}(\phi_1-\phi_2)(v_1-v_2)+\frac{1}{5L}(\phi_1-\phi_2)(2\psi_1+\psi_2)+\frac{2}{5L}\alpha_2(v_1-v_2)-\frac{1}{3}\alpha_2\psi_2-\frac{1}{15}\alpha_2\psi_1-\frac{1}{30}\alpha_1(\psi_1-\psi_2)\right\}+ \\
c_y\left\{\frac{6}{5L^2}(v_1-v_2)(\phi_1-\phi_2)+\frac{1}{5L}\left[(v_1-v_2)(2\alpha_1-\alpha_2)-(\phi_1-\phi_2)(2\psi_1-\psi_2)\right]-\frac{1}{3}\psi_2\alpha_2+\frac{1}{15}\psi_1\alpha_1+\frac{1}{30}(\psi_1\alpha_2+\psi_2\alpha_1)\right\} \\[2ex]
-c_x\left\{\frac{3}{5L^2}\left[(u_1-u_2)(\alpha_1-\alpha_2)+(\phi_1-\phi_2)(\theta_1-\theta_2)\right]-\frac{2}{5L}(\theta_1\alpha_1-\theta_2\alpha_2)\right\}- \\
c_y\left\{\frac{1}{5L^2}\left[-6(\phi_1-\phi_2)(\theta_1-\theta_2)+3(\alpha_1-\alpha_2)(u_1-u_2)\right]+\frac{1}{5L}\left[\alpha_1(\theta_1+2\theta_2)-\alpha_2(\theta_2+2\theta_1)\right]\right\} \\[2ex]
c_x\left\{-\frac{6}{5L^2}(u_1-u_2)(\phi_1-\phi_2)-\frac{1}{5L}\left[(u_1-u_2)(2\alpha_1-\alpha_2)+(\phi_1-\phi_2)(2\theta_1-\theta_2)\right]-\frac{1}{3}\theta_2\alpha_2+\frac{1}{15}\theta_1\alpha_1+\frac{1}{30}(\theta_1\alpha_2+\theta_2\alpha_1)\right\}+ \\
c_y\left\{\frac{3}{5L^2}(\phi_1-\phi_2)(u_1-u_2)+\frac{1}{5L}(\phi_1-\phi_2)(2\theta_1+\theta_2)-\frac{2}{5L}\alpha_2(u_1-u_2)-\frac{1}{3}\alpha_2\theta_2-\frac{1}{15}\alpha_2\theta_1-\frac{1}{30}\alpha_1(\theta_1-\theta_2)\right\} \\[2ex]
-c_x\left\{\frac{6}{5L^2}(u_1-u_2)(\psi_1-\psi_2)+\frac{3}{5L^2}(\theta_1-\theta_2)(v_1-v_2)-\frac{1}{5L}\theta_1(\psi_1+2\psi_2)+\frac{1}{5L}\theta_2(\psi_2+2\psi_1)\right\}- \\
c_y\left\{-\frac{6}{5L^2}(v_1-v_2)(\theta_1-\theta_2)-\frac{3}{5L^2}(\psi_1-\psi_2)(u_1-u_2)-\frac{1}{5L}\psi_1(\theta_1+2\theta_2)+\frac{1}{5L}\psi_2(\theta_2+2\theta_1)\right\} \\[2ex]
c_x\left\{-\frac{3}{5L^2}(u_1-u_2)(v_1-v_2)+\frac{1}{5L}(u_1-u_2)(2\psi_1+\psi_2)+\frac{2}{5L}\theta_2(v_1-v_2)-\frac{1}{3}\theta_2\psi_2-\frac{1}{15}\theta_2\psi_1-\frac{1}{30}\theta_1(\psi_1-\psi_2)\right\}+ \\
c_y\left\{-\frac{3}{5L^2}(u_1-u_2)(v_1-v_2)-\frac{1}{5L}(v_1-v_2)(2\theta_1+\theta_2)-\frac{2}{5L}\psi_2(u_1-u_2)-\frac{1}{3}\psi_2\theta_2-\frac{1}{15}\psi_2\theta_1-\frac{1}{30}\psi_1(\theta_1-\theta_2)\right\} \\[2ex]
\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array} \right\} \quad \text{(A8)}
$$

SOLUTION OF NONLINEAR PROBLEM

The load displacement relationships (Equation A5) were rewritten in the form

$$[K_E]\ \{q\} = \{Q\} - \{Q_A\} - \{Q_T\} \tag{A9}$$

The nonlinear terms were then included in the load terms as fictitious loads. The solution procedure involved adding the load in increments and at each load level solving for the nonlinear terms through an iterative procedure until the system was balanced. Then the next load increment was added and iterations performed until the system balanced. This procedure is costly and time-comsuming. As a result, only a few cases were actually run.

As was mentioned in the body of the report, the lateral buckling problem should lend itself to an eigenvalue analysis. If so, the critical buckling loads could be obtained with much less computer time. It may also be possible to consider the general loading problem and a combination buckling made in the form of an eigenvalue analysis. However, this required more time to develop than was available on this project. It would form an interesting topic to consider for future study. And, this analysis could yield useful results for use in evaluating the present AASHO requirements regarding critical stresses in overhead sign bridge supports.