

# **Modeling Reliability in Pavements**

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## ABSTRACT

The correct application of reliability to pavement design is essential to the objectives of pavement design which are to produce quality pavements to serve the traveling public in comfort and safety, being built to be durable in service at a minimum life cycle cost. Reliability is a technical term which is defined in mathematical terms and is therefore objective in its application. As applied to pavements, it makes use of either empirical or mechanistic pavement performance equations to predict an expected value and variance of either the traffic or the distress for which the pavement must be designed. Explicit expressions are found for the quantities in terms of the expected values and coefficients of variation (cv's) of the independent variables which appear in the performance equation(s). Although ample data are available to determine these for all pavement independent variables, experienced engineers can estimate realistic values of the cv's almost as well as they can be measured.

Several examples of the application of reliability to pavement performance models are given in this paper including:

1. Proper methods of analyzing field data to develop an empirical equation for the number of load cycles to reach pavement failure due cracking.
2. Pavement performance equations for cracking which incorporate reliability in determining the number of load cycles for which the pavement should be designed.
3. Relations of reliability to construct specifications.
4. Methods of incorporating reliability into the design of overlays while taking into account the variability of the cracking that

occurred in the old pavement.

In all cases, the mathematical nature of reliability is demonstrated. It is because of its ability to be defined in precise, unambiguous terms that makes it simple to apply and possible to set objective standards for the performance of pavements while taking into account the variability of the factors which control the performance. Reliability must be applied correctly in designing pavements in order to achieve its objective of providing an acceptable level of risk that pavements will carry the traveling public in comfort and safety while being built durably at minimum life cycle costs.

## Modeling Reliability in Pavements

### INTRODUCTION

The object of any pavement design procedure is to produce quality pavements which will perform according to expectations of the design engineer and the traveling public. These expectations must be achieved by simultaneously considering the paving materials to be used and their behavior under different traffic and environmental conditions many variables are involved in predicting the performance of a pavement, which is essential to the design process and they include the traffic mix and growth rates, rehabilitation timing, subgrade conditions, construction and aggregate sources and characteristics, material strengths, and weather among others.

Whether a new pavement or a rehabilitated pavement is being designed, it must be stated plainly that what is being designed does not yet exist. Design is the process whereby the variability and uncertainty that exists in the factors that will control the performance of the pavement are taken into account in providing an acceptable level of risk that the pavement will meet its performance expectations. This is what reliability is all about.

Reliability is defined simply as the probability that something will not fail. To put it another way, reliability is 1.0 minus the risk of failure. This definition of reliability introduces several terms warranting definitions: failure, probability, and risk.

Failure. Failure in pavements occurs when it can no longer function as it was intended. Pavements are intended to serve the traveling public in comfort and safety in the most economic manner possible, being built to be durable in service at a minimum life cycle cost. Translated into

technical terms, "comfort" is diminished by pavement roughness; "safety" is decreased by rutting (in asphalt pavements) and loss of surface friction; "durability" is reduced by the premature need to repair, rehabilitate, or reconstruct the pavement due to distress of any type; and "life cycle costs" include both initial construction costs, subsequent maintenance and rehabilitation costs, and user costs associated with user time delay costs, and excess user vehicle operating costs caused by the repair operation. "Failure" is then defined technically as the appearance of a pavement condition which requires some form of maintenance or rehabilitation. Thus, any of the following constitute failure of a pavement:

- (a) fatigue (alligator) cracking of an asphaltic concrete pavement above an acceptable limit in area or severity;
- (b) rutting depth of an asphaltic concrete pavement greater than an amount required to promote hydroplaning under wet weather conditions;
- (c) excessive fatigue cracking of a concrete pavement;
- (d) faulting of a jointed or transversely cracked concrete pavement above a level that produces a rough ride or the likelihood of rapid deterioration of the joints or cracks;
- (e) spalling of a concrete pavement which approaches a safety hazard, a rough ride, or a rapidly deteriorating pavement;
- (f) surface friction coefficient dropping below the level required to reduce wet weather accident rates to an acceptable minimum;
- (g) roughness of the pavement rising above a level that produces an unacceptably rough ride to the traveling public;
- (h) the appearance of any other condition that requires repair and traffic delay.

Probability. Probability is a number between 0 and 1 indicating a degree of likelihood of the occurrence of an event: 0 indicates that it will never occur and 1.0 indicates that it will certainly occur. Numbers in between 0 and 1 are measures of the likelihood of such an occurrence.

Risk. The adequacy of a pavement design is generally determined by assessing the risk associated with the ability of the pavement system to accommodate the imposed demands implemented by the highway agency, the public (the user), and the environment. Accepted levels of accuracy, of reliability, or of risk, are realistically subject to the balance of economic judgements and user benefits to be realized. In highway pavement design, the latter plays a greater role because of the more direct impact upon the traveling public. The need for a deterministic method that describes the level of adequacy in highway design has been well recognized for several years.

"So that designers can better evaluate the reliability of a particular design, it is necessary to ... predict variations in the pavement system response due to statistical variations in the input variables, such as load, environment, pavement geometry, and material properties including the effects of construction and testing variables." [1]

Several definitions of design reliability have been provided in recent highway research literature in terms of probabilistic concepts. An example is in the AASHTO design guide [2]:

"Reliability is the probability that the pavement system will perform its intended function over its design life and under the conditions (or environment) encountered during operation."

The same source also defines reliability with respect to specific types of pavement distress:

"Reliability is the probability that any particular type of distress will remain below or within a permissible level (as defined by the design engineer) during the design life." [2]

The definition of reliability in the textbook by Harr is as follows:

"Reliability is the probability of an object (an item as a system) performing its required function adequately for a specified period of time under stated conditions. This definition contains four essential elements:

1. Reliability is expressed as a probability
2. A quality of performance is expected
3. It is expected for a period of time
4. It is expected to perform under specified conditions". [3]

This definition of reliability applied to pavements, means the probability that it will not fail to perform any of its intended functions. These functions, as noted above, pertain for an extended analysis period which includes both the initial and subsequent performance periods under the in-service conditions of expected traffic and weather. Risk is the probability of failure in this function and is related to the reliability, since the sum of the two must always equal 1.0.

To quote again from the book by Harr [3], "Failure is highly qualitative and subjective; reliability, on the other hand, can be defined, quantified, tested, and confirmed."

Thus, design reliability is not subjective or qualitative, but necessarily mathematical and based upon mathematical principles and fundamentals. The definitions are rigorous and incontrovertible; they are not a subjective matter of opinion. The fundamentals and concepts in this paper can be found in professional journals, textbooks, and reference works on the subject of probability, statistics, and reliability. Some of this applications to pavements are original with this paper.

The application of reliability to pavements makes it possible to set objective standards of performance and to provide for the selection of pavements which will best serve their intended functions of carrying the traveling public in comfort and safety while providing this service with durable materials placed and maintained at the least life cycle costs. Reliability must be applied correctly to pavements in order to achieve this

variables. He also showed examples of the number of simulations to achieve a particular level of reliability. It is pointed out that the number of simulations or trials increase in proportion to the power to the number of independent variables.

Several approximations are used for the expected values and variances, including Cornell's approximations which are:

$$E[g(x)] \cong g(\mu) \quad (1)$$

$$\text{Var}[g(x)] \cong [g'(\mu)]^2 \text{Var}(x) \quad (2)$$

$$E[g(x,y)] \cong g(\mu_x, \mu_y) \quad (3)$$

$$\text{Var}[g(x,y)] \cong g_x^2(\mu_x, \mu_y) \text{Var}(x) + g_y^2(\mu_x, \mu_y) \text{Var}(y) + 2g_x g_y \text{cov}(x, y) \quad (4)$$

This is also called the first order, second moment (FOSM) approximation.

The Rosenbleuth method [4] does not make use of the Taylor series but instead uses calculated values of the function of  $x$  and  $y$  at different points, with each calculated value weighted to produce estimates of the expected value and variance of the function. This method is known as the Point Estimate Method (PEM) [4] and is useful from the standpoint that the existence and continuity of the first and second derivatives of the function  $f(x)$  is not a strict requirement. Point estimates of the function (for one variable) can be used in an expression for the expected value:

$$E(g(x)) = \frac{1}{2} (g(x)_+ + g(x)_-) \quad (5)$$

This expression expanded to multiple variables ( $m$ ) [13]:

$$E(g(x)) = \frac{1}{2^m} (g(x)_{\dots\dots\dots} + g(x)_{\dots\dots\dots} + g(x)_{\dots\dots\dots} + g(x)_{\dots\dots\dots}) \quad (6)$$



objective, and hence its ultimate importance. Some definitions of terms used in reliability are given in Appendix I.

## METHODS OF RELIABILITY ANALYSIS

Several methods of reliability have been developed which provide the distribution of random variables. These methods can be categorized into three types.

1. Exact methods,
2. First order, second-moment (FOSM) method, and
3. Point Estimate Method (PEM)

Each of these methods have distinctive characteristics and assumptions. The exact methods are usually computer-oriented and require several numerical integrations. In these methods, the probability distribution function must be known for each of the component (independent) variables associated with a random variable function. In some instances, the unknown component distributions can be assumed to be normal, log normal, or uniform. More is given about the selection of probability distributions subsequently. Numerical integration and Monte Carlo methods are included in this category. Although mainframe computers are mandatory, the main advantage of these methods is that they provide a complete probability distribution of the random, dependent variables. The disadvantage is that the output may be no better than the inputs (if they are assumed) and that considerable computer time is involved. In particular, methods such as the Monte Carlo method simulates random values or trials of the variable (associated with some probability distribution) based on the generation of random number inputs for the independent variables. Harr [3] showed how the randomly generated inputs can be associated with the assumed distribution of the independent

The total number of variables involved in equation 14 are  $2^m$ . The values of the function  $f(x)$  at the + and - levels correspond to the mean value of the first variable  $x$  plus or minus the associated standard deviation of the  $x$  variable. The first variable at minus one standard deviation is used to evaluate the second term in Equation 6 while all other variables are at plus one standard deviation. Numerous other methods are in use to estimate the expected values and variances of functions, but they will not be covered here. Because they are found in most textbooks and reference books on the subject of reliability.

#### SELECTION OF PROBABILITY DISTRIBUTIONS

The choice of probability distribution may be a function of mathematical convenience, such as the normal or log-normal distributions. These models may be selected even when no clear basis is provided. However, the type of probability model has been suggested to be related to the state of knowledge available [3]. If nothing is known about a probability distribution, then the only information available may be that of experience. Experience may not be able to substantiate that two probabilities are different in which case they must be assumed to be equal. The level of information may be evaluated in terms of the principle of maximum entropy which states the selected probability distribution should reflect the maximum entropy subject to any additional constraints imposed by the available information. Table 1 itemizes maximum entropy probability distributions for a list of given constraints.

Table 1. Maximum Entropy Probability Distributions [3]

Given Constraints	Assigned Probability Distribution
$\int_a^b f(x) dx = 1$	Uniform
$\int_a^b f(x) dx = 1$ Expected value	Exponential
$\int_a^b f(x) dx = 1$ Expected value, standard deviation	Normal
$\int_a^b f(x) dx = 1$ Expected value, standard deviation, range (minimum and maximum values)	Beta
$\int_a^b f(x) dx = 1$ Mean occurrence rate between arrivals of independent events	Poisson

An assumed theoretical distribution may be verified or disproved statistically by goodness-of-fit tests mentioned earlier. The chi-square test for distribution is a method to test the validity of an assumed distribution model. This test method compares the observed frequencies  $n_1, n_2, n_3, \dots, n_k$  of  $k$  values of the variate to corresponding frequencies  $e_1, e_2, e_3, \dots, e_k$  of an assumed theoretical distribution. The degree of goodness of fit is evaluated from the distribution of:

$$\sum_{i=1}^k \frac{(n_i - e_i)^2}{e_i} \quad (7)$$

This quantity approaches the chi-square  $(\chi^2)$  distribution for  $k-1$  degrees of freedom as  $n$  approaches infinity. If this quantity is less than the value of  $C_{1-\alpha, f}$  taken from the appropriate  $(\chi^2)$  distribution with a cumulative probability of  $(1-\alpha)$  at a significance level  $\alpha$ , the assumed theoretical distribution is acceptable. Otherwise, the distribution cannot be validated at the given level of significance.

The Kolmogorov-Smirnov test for distribution is another goodness-of-fit test involving the comparison between the experimental cumulative frequency and an assumed theoretical distribution. If a large difference exists, then the assumed distribution is rejected. This method is employed by developing a stepwise cumulative frequency function  $(S_n(x))$  based on re-arrangement of the sample data in increasing order [5]:

$$S_n(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \frac{k}{n} & \text{for } x_k \leq x < x_{k+1} \\ 1 & \text{for } x \geq x_n \end{cases} \quad (8)$$

where

$$\begin{aligned} x_1, x_2, \dots, x_n &= \text{ordered values of sample data} \\ n &= \text{sample size} \end{aligned}$$

As illustrated in Figure 1, this test is based on the maximum difference  $(D_n)$  between  $S_n(x)$  and an assumed theoretical distribution function  $F(x)$  over the range of  $X$ . Therefore the value of  $D_n$  is

$$D_n = \max_x |F(x) - S_n(x)| \quad (9)$$

and is a random variable whose distribution depends on  $n$  and, for a significant level of  $\alpha$ , is compared to a critical value  $D_n^\alpha$ . This critical value is defined by:

$$P(D_n \leq D_n^a) = 1 - \alpha \quad (10)$$

If the observed  $D_n$  is less than the tabulated value of  $D_n^a$ , then the proposed theoretical distribution is assumed as acceptable for the given level of significance.

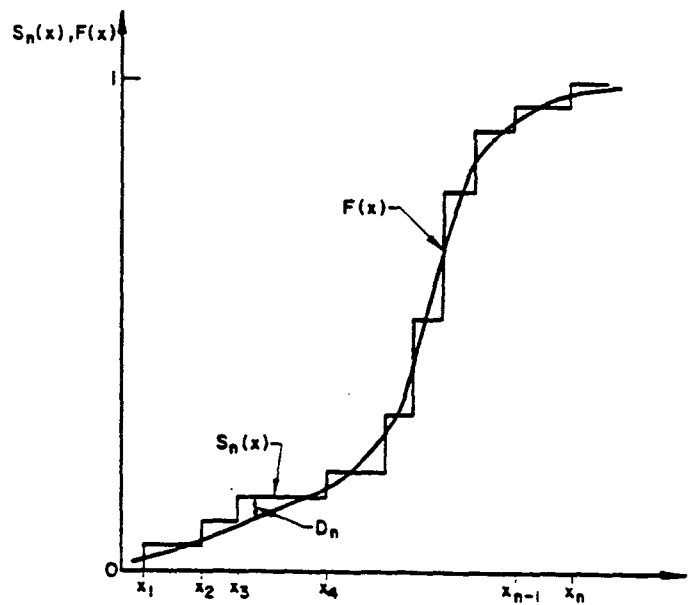


Figure 1. Empirical cumulative frequency vs. theoretical distribution function.

#### CHARACTERIZATION OF THE VARIABILITY OF INDEPENDENT VARIABLES IN PAVEMENT PERFORMANCE EQUATIONS

The variability observed in highway performance is primarily derived from variability in pavement materials, layer thicknesses, temperatures and moisture, in the frequency and magnitude of applied traffic loads and in the distribution of pavement distress along the length of the pavement. The

coefficient of variation (cv) serves as a measure of the dispersion of a particular variable and provides the basis by which individual variabilities can be included in the overall pavement reliability analysis (as suggested by the FOSM Method of reliability). The coefficient of variation of various civil engineering materials estimated as shown in Table 2 [3]. As is seen in Table 2, coefficients of variation of natural material properties vary between 2 and 30 percent. The 2 percent figure indicates a very narrow range and 30 percent is a very wide range. It is a common observation that experienced civil engineers can estimate the coefficient of variation of a material property with which they are familiar very closely, especially if they are familiar with the quality control exercised in the construction process. In the absence of any knowledge of the quality control, a maximum value of the cv of 30 percent will be a conservative assumption. Pavement layer materials are constructed, rather than natural, and typical values of their coefficients of variation for pavement materials are shown in Table 3 and in reference 6.

Once important parameters such as the expected value and variance (as a function of the various cv's) are determined the probability (P) of pavement failure, or the reliability (R) is found:

$$P = \text{Prob} [F(x_i) \geq F_0] = 1 - R \quad (11)$$

where

$F(x_i)$  = the functional relationship for a given pavement distress for a given period  $i$   
 $F_0$  = the failure condition for a given pavement distress

Table 2. Typical Coefficients of Variation [3]

Parameter	Coefficient of Variation, %		
Porosity	Soil	10	
Specific gravity		2	
Water content			
Silty clay		20	
Clay		13	
Degree of Saturation		10	
Unit weight		3	
Coefficient of permeability		(240 at 80% saturation to 90 at 100% saturation)	
Compressibility factor		16	
Preconsolidation pressure		19	
Compression index			
Sandy clay		26	
Clay		30	
Standard penetration test		26	
Standard cone test		37	
Friction angle $\phi$			
Gravel		7	
Sand		12	
c, strength parameter (cohesion)		40	
	Structural	Loads,	50-Year
Maximum			
Dead load		10	
Live load		25	
Snow load		26	
Wind load		37	
Earthquake load		>100	
	Structural Resistance		
Structural steel			
Tension members, limit state, yielding		11	
Tension members, limit state, tensile strength		11	
Compact beam, uniform moment		13	
Beam, column		15	
Plate, girders, flexure		12	
Concrete members			
Flexural strength, reinforced concrete, Grade 60		11	
Flexural strength, reinforced concrete, Grade 40		14	
Flexural, cast-in-place beams		8-9.5	
Short columns		12-16	
	Ice		
Thickness		17	
Flexural strength		20	
Crushing strength		13	
Flow velocity		33	
	Wood		
Moisture		3	
Density		4	
Compressive strength		19	
Flexural strength		19	
Glue-laminated beams			
Live load		18	
Snow load		18	

Table 3. Pavement Materials Coefficients of Variations

Property	Coefficient of Variation, %
Layer Thickness	
Surface	5 - 12
Base	10 - 15
Subbase	10 - 15
Elastic Modulus	
Surface	10 - 20
Base	8 - 20
Subbase	10 - 20
Subgrade	10 - 30
Fluid Content	
Surface - Asphalt	5 - 10
Surface - Water	5 - 10
Base - Water	10 - 15
Subbase - Water	10 - 15
Subgrade - Water	10 - 20
Density	
Surface	5 - 10
Base	10 - 15
Subbase	10 - 20
Subgrade	10 - 20
Tensile Strength	
Surface - Asphalt	10-15
Surface - Concrete	10-15
Base - Stabilized	10-15

The performance condition corresponding to the level of reliability is:

$$F_z = F(\bar{X}_i) - Z_R \sqrt{\text{Var} [F(x_i)]} \quad (12)$$

where

$F_z$  = the predicted performance condition reached with a probability  
1-R

$Z_R$  = value of the variate corresponding to the reliability level, R



It is obvious that the simple formulation provided above gives a straight forward and definable relationship between the mean level of performance and the performance at a given level of reliability. It is noted that this process of determining design reliability can be applied once the mean values and cv's of the variables used to predict the performance of the pavement are known or can be estimated as must always be done in design. The design process does not require the amassing of large quantities of field data over long periods of time before reliability can be used in design. Such data, when collected over time, can be used to refine the estimates of the means and cv's used in design and can be used to refine and calibrate the performance equations which are also used in design. This refinement process is a desirable use of these data, but is not a prerequisite for applying reliability in design. Instead, reliability can be computed using any pavement performance equation when estimates of the means and cv's of the independent variables are inserted into it. Examples of such equations which are presented below will illustrate this point.

#### **APPLICATION OF RELIABILITY**

The methods of applying reliability which are discussed in this section are applicable to any deterministic pavement performance model. The first order, second moment (FOSM) approach is used throughout this discussion, but it applies to any other formulation of reliability such as simulation or the point estimation method as well.

### FORMULATION OF RELIABILITY FOR DIFFERENT PROBABILITY DENSITY FUNCTIONS

The reliability factor,  $Z_R$ , may be computed for any probability density function by using the following steps:

1. Find the mean,  $\mu$ .
2. Find the variance,  $\sigma^2$ .
3. Find the area beneath the probability density function, from its lower limit, to the "maximum acceptable" upper limit,  $x_{\max}$ . This area is the reliability level,  $R$ .
4. Calculate the reliability factor,  $Z_R$ , from

$$Z_R = \frac{x_{\max} - \mu}{\sigma} \quad (13)$$

This process is illustrated in the graph below. The reliability factor can be computed for any probability density function,  $p(x)$ . The risk of failure is 1.0 minus the reliability.

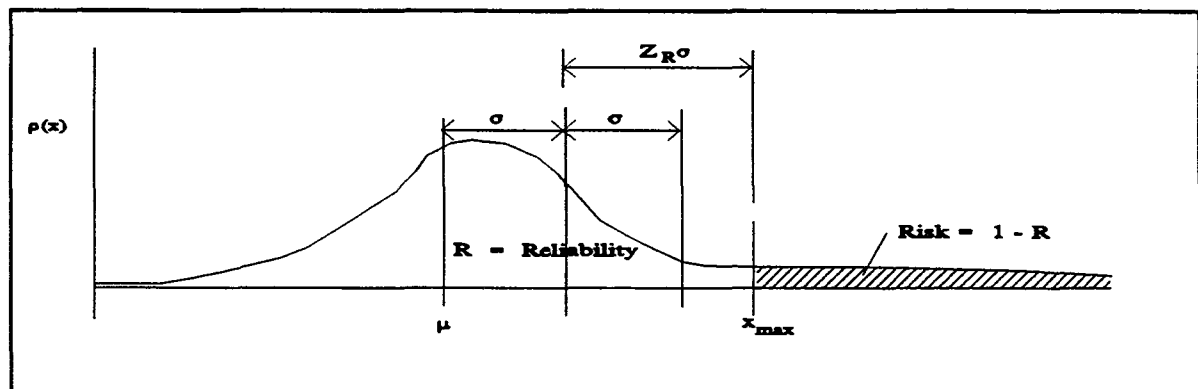


Figure 2. Reliability Factor,  $Z_R$ .

This process is straight forward when the PDF,  $p(x)$ , is a normal distribution. The mean and standard deviation are  $\mu$  and  $\sigma$ , respectively. With the log normal distribution and the Weibull distribution, expressions for the mean and standard deviation have been derived and are discussed in

Appendix II and are found in reference books. The known results for these distributions are given below for "damage",  $D$ , which is defined as in the AASHTO design equation, as a ratio of some level of distress to its maximum acceptable value.

$$D = \frac{\text{distress level}}{\text{maximum acceptable distress level}}$$

### Normal

$$x = \frac{D - \bar{D}}{\sigma_D}$$

### Log Normal

$$x = \frac{\ln D - E[\ln D]}{\sigma(\ln D)}$$

$$E[\ln D] = \lambda$$

$$\text{var}[\ln D] = \gamma^2$$

$$E(D) = \bar{D} = \exp\left[\lambda + \frac{\gamma^2}{2}\right]$$

$$\text{var}(D) = \sigma^2(D) = \exp[2\lambda + \gamma^2][\exp(\gamma^2) - 1]$$

$$cv^2(D) = \frac{\text{var}(D)}{\bar{D}^2} \cong \text{var}[\ln D]$$

### Weibull

$$x = \bar{D}$$

$$\bar{D} = E(x) = E(D) = \frac{1}{\lambda} \left[ \Gamma\left(1 + \frac{1}{\gamma}\right) \right]$$

$$\text{var}(x) = \text{var}(D) = \frac{1}{\lambda^2} \left[ \Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right) \right]$$

$\lambda$  = the Weibull scale parameter

$\gamma$  = the Weibull shape parameter

$\Gamma( )$  = the Gamma function

With both the log normal and Weibull distributions, if the expected value and variance of the damage is known, then both the scale and shape parameters may be found.

#### ANALYSIS OF NUMBER OF LOAD APPLICATIONS TO REACH FAILURE, $N_f$

Applying what is known concerning pavements also applies to the form of the equation which defines the relation between cracking and the number of load applications. It is known that cracking does not occur at the same time over the entire length of the pavement. It is also known that it does not occur uniformly at all locations along a pavement section of uniform construction. Thus, it is known and has been represented in mechanics as the result of a stochastic process. Analysis of the cracking behavior of a pavement as a function of estimated traffic, if it is to respect what is known of its behavior, must make use of the forms of equation that are used in probability.

The question of which form of equation to use may be posed by asking which of the relations (a), (b), or (c) in Figure 3 should be used to relate the expected value of cracking ( $\bar{C}$ ) to the expected value of the traffic load application ( $\bar{N}$ ).

With cracking data, one is faced with analyzing the relationship between two probabilistic quantities, traffic and cracking. The recorded traffic is an estimate of the actual traffic and, as with all traffic

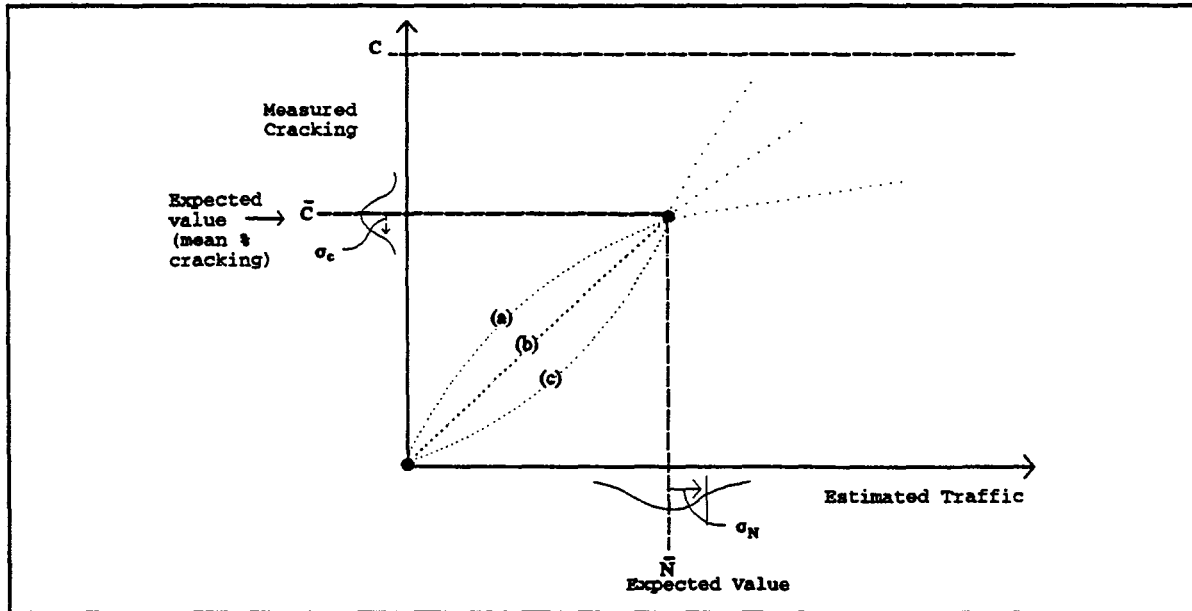


Figure 3. Possible Relations Between  $c$  and  $N$  for a Single Pavement Section.

estimates, has an expected value ( $\bar{N}$ ) as its most likely value, and a likely range within which the actual value will fall. Thus, the estimate of  $N$ , the number of traffic load applications, has a probability density function (PDF) which can be characterized by its mean ( $\mu$ ) its standard deviation ( $\sigma$ ) and its mathematical form. Commonly used mathematical forms used with traffic data are normal, log normal, and Poisson.

The recorded cracking is also a mean value ( $\bar{c}$ ) measured over an entire pavement section, although the occurrence of cracking in the section is by no means uniformly distributed. Instead,  $\bar{c}$  is arrived at by measuring all of the cracking along the pavement section and dividing by the total area of pavement that could be cracked as a maximum. Thus, the recorded value of cracking ( $\bar{c}$ ) is also an estimate of the expected value and represents a range of values which are likely to occur on the pavement from point to

point. This indicates that cracking also has a probability density function which is represented by a mean and a standard deviation and its mathematical form. Commonly used forms of equations used to describe cracking frequency are the log normal, Weibull, and Gumbel.

This now leads to the question of how  $\bar{c}$  and  $\bar{N}$  are related to one another. In the first place, it is recognized that this relation, whatever it is, is unique to the pavement on which it is measured. Secondly, it is recognized that the value of  $\bar{c}$  has absolute limits of 0 and 1. Any mathematical form of the relation between  $\bar{c}$  and  $\bar{N}$  that allows  $\bar{c}$  to go below 0 or to go above 1 is automatically invalid. Thus the relations:

(24)

and

(25)

which are illustrated in Figure 3 as curves (b) and (c) are inappropriate mathematical forms to use in describing the relationship between  $\bar{c}$  and  $\bar{N}$ .

On the other hand, an appropriate mathematical relation as illustrated in Figure 3, curve (a). The form of this equation is in accordance with viscoelastic fracture mechanics and is:

$$\bar{c} = \text{Prob} [\text{Damage} > 1.0] \quad (26)$$

where

$$\text{Damage} = \sum_{i=1}^k \frac{n_i}{N_{fi}} \quad (27)$$

$n_i$  = the number of load applications of load level,  $i$

$N_{fi}$  = the number of load applications of load level,  $i$ , to cause failure.

If equivalent load applications are used in the traffic estimates, then the damage equation simplifies to:

$$\text{Damage} = \frac{N}{N_{fi}} \quad (28)$$

When the damage ratio equals 1.0, at a given spot on the road surface, the crack, which has been working its way through the pavement, appears on the surface. The relation between the mean values of cracking and traffic load applications is governed by the probabilistic relationship that the mean value of cracking ( $\bar{c}$ ) is related to the traffic estimate ( $N$ ) by the probability that the damage ratio,  $N/N_{fi}$ , is equal to 1.0. That is:

$$\bar{c} = \int_1^{\infty} p(x) dx \quad (29)$$

where

$$x = \frac{D - \bar{D}}{\sigma_D} \quad \text{for a normal distribution}$$

$$D = \text{damage} = \frac{N}{N_{fi}}$$

$$\bar{D} = \text{mean value of damage corresponding to } \bar{N}$$

$$\sigma_D = \text{the standard deviation of damage}$$

$$p(x) = \text{the probability density function that is appropriate for cracking. This is, typically, a log normal, Gumbel, or Weibull distribution.}$$

The definition of  $x$  can be defined in terms of  $N$  and  $N_{fi}$  as follows (see Appendix III):

$$x = \frac{N - \bar{N}}{\sigma_N \left[ 1 + \frac{cv^2(N_{fi})}{cv^2(N)} \right]^{\frac{1}{2}}}$$

$$dx = \frac{dN}{\sigma_N \left[ 1 + \frac{cv^2(N_{fi})}{cv^2(N)} \right]^{\frac{1}{2}}}$$

The probability density functions that are appropriate for this relation are described below (described in Appendix II):

Gumbel (Type II Asymptotic Form):

$$p(x) = \beta \rho^\beta x^{-(1+\beta)} e^{-\left(\frac{\rho}{x}\right)^\beta} \quad (32a)$$

$$= \beta \rho^\beta x^{-(1+\beta)} \exp \left[ -\left(\frac{\rho}{x}\right)^\beta \right] \quad (32b)$$

where

$$\begin{aligned} \beta &= \text{shape parameter (k)} \\ \rho &= \text{scale parameter (v}_n\text{)} \\ x &= \text{damage, } D = N/N_f \end{aligned}$$

The cumulative probability distribution function,  $P(x)$ , which corresponds to the Gumbel PDF and fits through the data points relating  $\bar{c}$  and  $N$  is:

$$\bar{c} = P(x) = \exp \left[ -\left\{\frac{\rho}{x}\right\}^\beta \right] \quad (33)$$

$$\text{and } \bar{x} = \bar{D} \cong \frac{\bar{N}}{N_f}$$

Weibull:

$$p(x) = \gamma \lambda (\lambda x)^{\gamma-1} \exp - \{\lambda x\}^\gamma$$

where

$$\begin{aligned} \gamma &= \text{shape parameter (}\beta\text{)} \\ \lambda &= \text{scale parameter (}1/\alpha\text{)} \end{aligned}$$

The cumulative probability distribution function,  $P(x)$ , which corresponds to the Weibull PDF is the following exponential function, which relates to  $\bar{c}$  to  $\bar{N}$  for each pavement section:

$$\bar{c} = P(x) = 1 - \exp \left[ -(\lambda \bar{x})^\gamma \right] \quad (34a)$$

$$\text{and } \bar{x} = \bar{D} \cong \frac{\bar{N}}{N_f} \quad (34b)$$



These are the functions that should be used to fit the data recorded for each pavement section, one section at a time. Under most circumstances, only two points are known for each pavement section:

1. The origin where  $\bar{N} = 1$  and the cracking is an assumed small value, say 0.001.
2. The actual measured point,  $\bar{c}$  and  $\bar{N}$  (  $x$  represents  $N$  ).

The scale and shape parameters can be found for each pavement section using these two points on the curve, either the cumulative Gumbel or the Weibull exponential function. The two parameters, in turn, can be seen to depend upon the following:

1. The current value of the expected value of the number of traffic load applications,  $N$ .
2. The variance of the traffic estimate,  $\text{Var}(N)$ , (or standard deviation  $\sigma_N = [\text{Var}(N)]^{1/2}$ ) and the coefficient of variation (cv) of  $N$ , which is  $[\text{Var}(N)/N^2]^{1/2}$ .
3. The coefficient of variation of the number of load cycles to reach failure,  $\text{cv}(N_f)$ .

It is possible, then, taking one pavement section at a time to determine the value of  $\bar{N}$  corresponding to some pre-set value of  $\bar{c}$  corresponding to some pre-set value of  $\bar{c}_{\max}$  which is determined to be a maximum acceptable value. This is the value of  $\bar{N}_f$  which is included in the definition of damage (See Appendix A).

With the Gumbel cumulative distribution, the scale and shape factors,  $\rho$  and  $B$ , can be determined by linear regression, one pavement section at a time using the equation:

$$y_i = a + b x_i \quad (35)$$

where

$$y_i = \ln [-\ln \bar{c}_i]$$

$$x_i = \ln \bar{N}_i$$

$$b = -\beta$$

$$a = \beta \ln \rho$$

$$\rho = \exp \left\{ -\frac{a}{b} \right\}$$

Once these factors are known, the value of  $\bar{N}_f$  can be found using the following formula:

$$\bar{N}_f = \frac{\rho}{\left[ -\ln (\bar{c}_{max}) \right]^{\frac{1}{\beta}}} \quad (36)$$

Values of  $\bar{N}_f$  can be determined for each pavement by this means. The value  $\bar{N}_f$  that is determined is a unique property of each individual pavement section, and is a value by which each pavement may be compared because it represents the number of load applications at which a standard condition of distress of each pavement is reached.

By similar means, the scale and shape factors for the Weibull exponential function may be formed, by using a linear regression technique and the two known points on the curve, using the following equation:

$$y_i = a + b x_i \quad (37)$$

where

$$\begin{aligned} y_i &= \ln [-\ln (1 - \bar{c}_i)] \\ b &= \gamma \\ a &= \gamma \ln \lambda \\ \lambda &= \frac{\exp \{a/b\}}{\gamma} \\ x_i &= \bar{N}_i \end{aligned}$$

Once these coefficients are known, the value of  $\bar{N}_f$  can be found using the following formula:

$$\bar{N}_f = \frac{1}{\lambda} \left[ -\ln (1 - \bar{c}_{\max}) \right]^{\frac{1}{\nu}} \quad (38)$$

The value of  $\bar{N}_f$  that is determined in this way may be used as the dependent variable in an expression that represents the number of load cycles to reach failure as derived from fracture mechanics.

The log normal probability density function defines  $x$  in a slightly different way, as follows:

$$x = \frac{\ln D - E[\ln D]}{\sigma (\ln D)} \quad (39)$$

where

$$D = \frac{N}{N_f}$$

$$E[D] = \bar{D} = \exp \left[ \lambda + \frac{\zeta^2}{2} \right]$$

$$\lambda = E[\ln D]$$

$$\zeta^2 = \text{Var} [\ln D]$$

$$cv^2 (D) = \frac{\text{Var}(D)}{\bar{D}^2} \cong \text{Var}[\ln D]$$

$$\sigma^2(D) = \text{Var}(D) = \exp[2\lambda + \zeta^2] [\exp(\zeta^2) - 1]$$

The probability density functions of the damage relation is:

$$p(x) = \frac{1}{\zeta D \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x}{\zeta} \right)^2 \right] \quad (40)$$

The cumulative distribution function,  $p(x)$ , which fits through the points relating  $\bar{c}$  to  $\bar{N}$  is given by:

$$\bar{c} = \int_{\frac{-\ln D}{\sigma(\ln D)}}^{\infty} p(x) dx \quad (41)$$

The expected value of the damage,  $D$ , is approximated by:

$$E(D) = \bar{D} \cong \frac{\bar{N}}{N_f} \quad (42)$$

The variance of the damage,  $D$ , is approximated by:

$$\text{Var}(D) \cong \bar{D}^2 [cv^2(N) + cv^2(N_f)] \quad (43)$$

which may also be written as:

$$\sigma(D) \cong \bar{D} cv(N) \left[ 1 + \frac{cv^2(N_f)}{cv^2(N)} \right]^{\frac{1}{2}} \quad (44)$$

The determination of the two constants,  $\lambda$  and  $\zeta$ , can be accomplished using non-linear regression by minimizing the sum of squared errors.

$$\sum \epsilon_i^2 = \sum_{i=1}^2 \left[ \bar{c}_i - \int_{\frac{-\ln D_i}{\sigma(\ln D)}}^{\infty} p(x, \lambda, \zeta) dx \right]^2 \quad (45)$$

for each pavement. The two known pairs of points are:

$\bar{c}$	$\bar{N}$
$\epsilon$	1
$\bar{c}$	$\bar{N}$

and the value of  $\epsilon$  is set at a small value around 0.001. The value of  $N_f$  can be found by the same trial-and-error process by finding the value of  $N_f$  that produces a value of the area of cracking equal to  $c_{max}$ .

A common, but mistaken, approach to the characterization of the cracking of pavements is to plot a graph of the  $(\bar{c} - \bar{N})$  pairs from a number

of pavements and find two coefficients by regression analysis which fit through the cloud of data points.

A curve of the form:

$$\bar{c} = a (\bar{N})^b \quad (46)$$

is assumed and the mean-squared error of this is assumed erroneously to be the standard deviation of the cracking (PDF (a) in Figure 4) or the standard deviation of the traffic,  $\bar{N}$  (PDF (b) in Figure 4) at a set value of  $\bar{c}$  is assumed erroneously to be the standard deviation of the number of load applications at that level of cracking.

It is recalled that this graph is a graph of the means of both cracking and load applications. The standard deviation of the means of sampled populations is known not to be equal to the standard deviation of the population. In fact, if the probability density function of  $N$  were normal, the relation between the standard deviation of  $N$  and the standard deviation of  $\bar{N}$  is known to be:

$$\sigma_{\bar{N}} = \frac{\sigma_N}{\sqrt{n}} \quad (47)$$

where

- $n$  = the number of pavements sampled
- $\sigma_{\bar{N}}$  = the standard deviation of the means, as it can be derived from a plot of  $\bar{c}$  vs  $\bar{N}$ .
- $\sigma_N$  = the standard deviation of the actual traffic load applications.

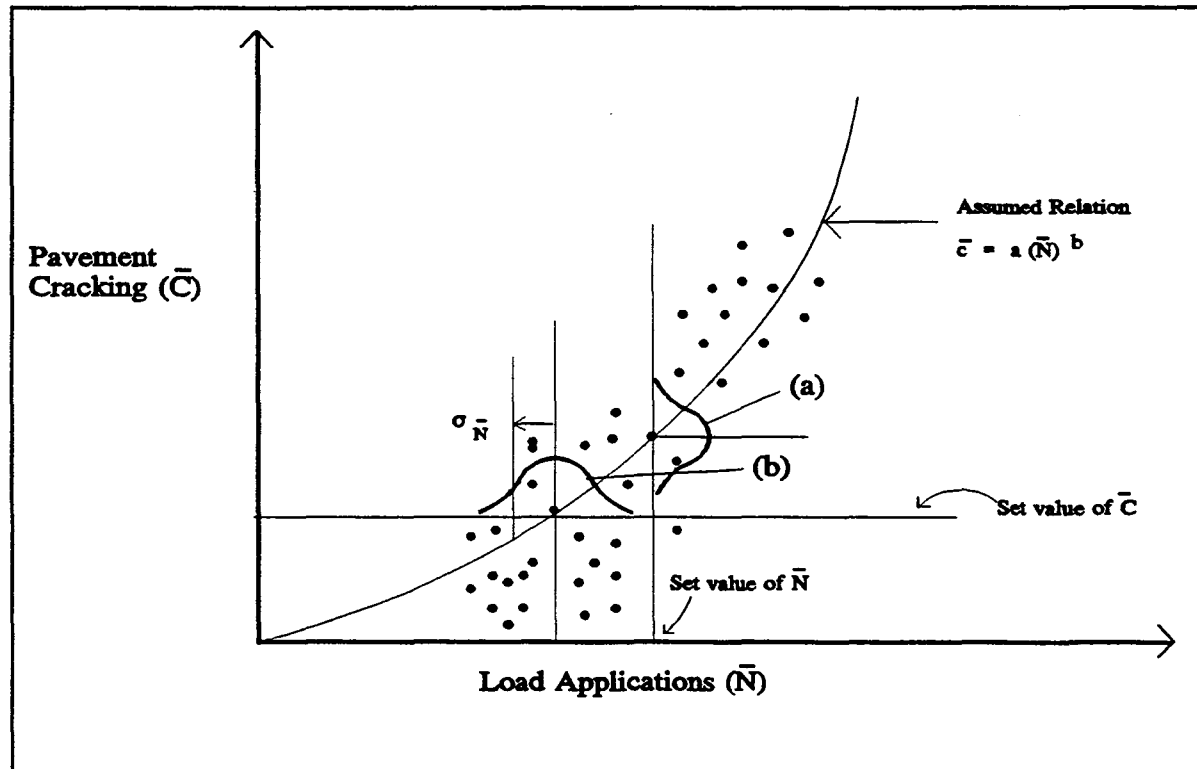


Figure 4. Pavement cracking versus load application.

Thus, if  $\sigma_{\bar{N}}$  is found to be, say 20%, of the mean traffic level,  $\bar{N}$ , from a population,  $n$ , of 36 pavements, then the standard deviation of the traffic,  $\sigma_N$ , is 120%, following from the relation given above. Although the

$$\sigma_N = \sqrt{n} \sigma_{\bar{N}} \quad (48)$$

relation is valid for a normal distribution, a similar relation applies to all probability density functions. Furthermore, the use of  $\sigma_{\bar{N}}$  to estimate the reliability of a pavement design is incorrect, no matter what the probability density function of the actual traffic. As referred to in Chapter 5 and stated plainly here, there is no way that the standard deviation of either the traffic estimate ( $\sigma_N$ ) or the cracking area ( $\sigma_c$ ) can

be determined from collections of measured expected values of cracking ( $\bar{C}$ ) and estimated traffic ( $\bar{N}$ ). Any standard deviation derived from such data will always be too small by a large multiplying factor.

#### DETERMINATION OF THE NUMBER OF LOAD CYCLES TO REACH FAILURE, $N_f$

The number of load applications to reach a specific level of cracking can be determined by the analysis of field data, as described before. This same number of load cycles to reach a specified level of cracking,  $N_f$ , can be determined by the use of laboratory measurements and analytical calculations. One such method of relating the field value of  $N_f$  to the calculated value of the wheel load strain at the bottom of the asphalt layer is called the "phenomenological approach." A similar method is used to predict the number of load cycles to reach a specified level of cracking in concrete pavements, and the relation is between  $N_f$  and the wheel load stress at the bottom of the concrete layer.

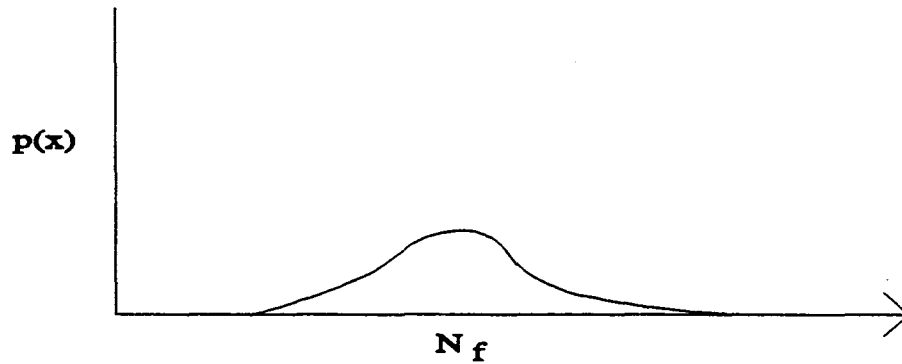
In the asphalt pavement, the phenomenological equation is:

$$N_f = K_1 \left( \frac{1}{\epsilon} \right)^{K_2} \quad (49)$$

In the concrete pavement, the equation is of the form:

$$N_f = K_1 \left( \frac{1}{\sigma} \right)^{K_2} \quad (50)$$

The strain,  $\epsilon$ , and stress,  $\sigma$ , in the above equations are calculated using elastic theory. As noted before,  $N_f$ , is not a single number for a given pavement but is distributed in a probabilistic way. The probability density function for  $N_f$  on a specific pavement is illustrated in Figure 5.

Figure 5. Distribution of  $N_f$ 

This is a result of the fact that the values of  $K_1$ ,  $K_2$ ,  $\epsilon$ , and  $\sigma$  in the above equations are also distributed over a range of values.

Because  $N_f$  is probabilistic,  $N_f$  has an expected value and a variance, as do  $K_1$  and  $K_2$ , and it has been found that  $K_1$  and  $K_2$  are correlated, so that there is a covariance of  $K_1$  and  $K_2$ . If  $N_f$  is normally distributed, the expected value of  $N_f$  for asphalt pavements is:

$$\bar{N}_f = E(N_f) \cong \bar{K}_1 \left[ \frac{1}{\epsilon} \right]^{K_2} \quad (51)$$

and its variance is (as shown previously assuming the lack-of-fit error ( $\epsilon$ ) = 0):

$$\begin{aligned} \text{Var}(N_f) = & \bar{N}_f^2 \text{cv}^2(k_1) \\ & + \bar{K}_2^2 \bar{N}_f^2 (\ln \bar{\epsilon})^2 \text{cv}^2(k_2) \\ & + \bar{K}_2^2 \bar{N}_f^2 \text{cv}^2(\epsilon) \\ & + 2 \bar{\epsilon}^{-k_2} \bar{N}_f \{ \rho_{k_1, k_2} \} k_1 k_2 \text{cv}(k_1) \text{cv}(k_2) \end{aligned} \quad (52)$$



where

- $\rho_{K_1, K_2}$  = the correlation coefficient between  $K_1$  and  $K_2$   
 $cv(K_1)$  = the coefficient of variation of  $K_1$   
 $cv(K_2)$  = the coefficient of variation of  $K_2$   
 $\bar{K}_1$  = the expected value of  $K_1$   
 $\bar{K}_2$  = the expected value of  $K_2$   
 $\bar{\epsilon}$  = the expected value of the calculated strain  
 $\bar{N}_f$  = the expected value of the number of load applications

The expected value for concrete pavements is:

$$E(N_f) = \bar{N}_f \cong \bar{K}_1 \left( \frac{1}{\sigma} \right)^{K_2} \quad (53)$$

and the variance is:

$$\begin{aligned}
 \text{Var}(N_f) = & \bar{N}_f^2 cv^2(K_1) \\
 & + \bar{K}_2^2 \bar{N}_f^2 (\ln \sigma)^2 cv^2(K_2) \\
 & + \bar{K}_2^2 \bar{N}_f^2 cv^2(\epsilon) \\
 & + 2 \bar{\sigma}^{-K_2} \bar{N}_f \{ \rho_{K_1, K_2} \} K_1 K_2 cv(K_1) cv(K_2)
 \end{aligned} \quad (54)$$

These are the expected values and variances that are to be used with the Gumbel and Weibull distributions as well. If the lack-of-fit error is not zero, its square is added to the variance term.

If  $N_f$  is log normally distributed the expected values and variances are computed in a different way, as follows. The natural logarithm is taken of both sides of the phenomenological equation for  $N_f$ . For asphalt pavements:

$$\ln N_f = \ln K_1 - K_2 \ln \epsilon \quad (55)$$

and for concrete pavements:

$$\ln N_f = \ln K_1 - K_2 \ln \sigma \quad (56)$$

The expected value of  $\ln N_f$  for asphalt pavements is:

$$E[\ln N_f] \cong \ln \bar{K}_1 - \bar{K}_2 \ln \bar{\epsilon} \quad (57)$$

The expected value of  $\ln N_f$  for concrete pavement is:

$$E [\ln N_f] \cong \ln \bar{K}_1 - \bar{K}_2 \ln \bar{\sigma} \quad (58)$$

The variance of  $\ln N_f$  for asphalt pavement is estimated by:

$$\text{var} [\ln N_f] \cong \text{cv}^2(K_1) + \bar{K}_2^2 (\ln \bar{\epsilon})^2 \text{cv}^2(K_2) - 2 \ln \bar{\epsilon} (\bar{K}_2) \rho_{K_1, K_2} \text{cv}(K_1) \text{cv}(K_2) \quad (59)$$

The variance of  $\ln N_f$  for concrete pavements is given by:

$$\text{var} [\ln N_f] \cong \text{cv}^2(K_1) + \bar{K}_2^2 (\ln \bar{\sigma})^2 \text{cv}^2(K_2) - (\ln \bar{\sigma}) (\bar{K}_2) \rho_{K_1, K_2} \text{cv}(K_1) (\text{cv} K_2) \quad (60)$$

Reliability may be defined in one of two ways as (1) the probability that the traffic will not exceed  $N_f$  or (2) the probability that the cracking area will not exceed the maximum acceptable level. Both of these approaches to reliability require that a probability density function for the variable of interest ( $N_f$  or area cracked) is specified.

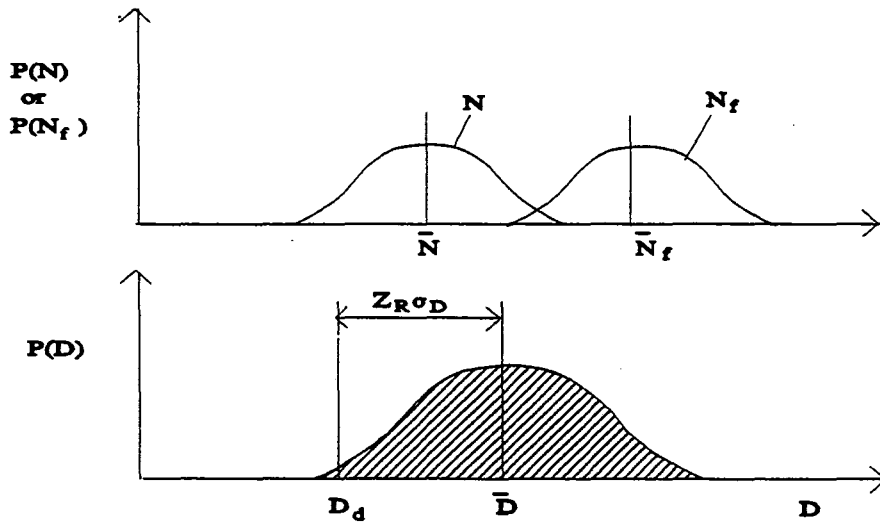
Reliability takes into account the fact that both the expected traffic,  $N$ , and the number of traffic load applications to reach failure,  $N_f$ , are probabilistic, each having expected values,  $\bar{N}$  and  $\bar{N}_f$ , and variances,  $\text{var} (N)$  and  $\text{var} (N_f)$ , respectively.

If both  $N$  and  $N_f$  are normally distributed, a difference distribution ( $D$ ), also normally distributed, may be defined. They are illustrated in Figure 6.

$$D = N_f - N \quad (61)$$

$$\bar{D} = E(D) = \bar{N}_f - \bar{N} \quad (62)$$

$$\text{Var} (D) = \text{var} (N_f) + \text{var} (N) \quad (63)$$

Figure 6. Difference Distribution,  $N_f - N$ .

Reliability may be defined in terms of the difference distribution. Reliability,  $R$ , is the probability that  $N_f$  is greater than  $N$ , i.e.

$$R = \text{Prob} [N_f > N] \quad (64)$$

This is the same as saying that reliability,  $R$ , is the probability that  $N_f - N$  is greater than 0.

$$R = \text{Prob} [(N_f - N) > 0] \quad (65)$$

Referring to the graph of the difference distribution, the distance  $D_d - \bar{D}$  is equal to  $Z_R \sigma_D$ , where  $D_d$  is the design minimum value of the difference distribution ( $D_d = 0$ ) and  $Z_R$  is the normal variable corresponding to the desired level of reliability.

$$D_d - \bar{D} = Z_R \sigma_D \quad (66)$$

$$D_d = 0 = \bar{D} + Z_R \sigma_D \quad (67)$$

$$0 = \bar{N}_f - \bar{N} + Z_R [\text{Var} (N_f) + \text{Var} (N)]^{1/2} \quad (68)$$

$$\bar{N}_f = \bar{N} - Z_R \left\{ \bar{N}_f^2 \text{cv}^2(N_f) + \bar{N}^2 \text{cv}^2(N) \right\}^{1/2}$$

$$\left\{ \frac{\bar{N}_f}{\bar{N}} - 1 \right\}^2 = Z_R^2 \left\{ \left( \frac{\bar{N}_f}{\bar{N}} \right)^2 \text{cv}^2(N_f) + \text{cv}^2(N) \right\}$$

After some algebra and use of the quadratic formula:

$$\bar{N}_f = \bar{N} - Z_R \bar{N} \left[ Z_R^2 \text{cv}^2(N) \text{cv}^2(N_f) - \text{cv}^2(N_f) - \text{cv}^2(N) \right]^{1/2} \left\{ 1 - Z_R^2 \text{cv}^2(N_f) \right\}^{-1} \quad (71)$$

The value of  $Z_R$  is taken from the list of normal variables corresponding to the desired level of reliability in Table 4.

Table 4. Reliability Factor,  $Z_R$ , Values.

Reliability Level, %	Normal Variable $Z_R$
50.0	-0.000
70.0	-0.525
80.0	-0.84
85.0	-1.04
90.0	-1.28
95.0	-1.64
99.0	-2.33
99.9	-3.08

If both  $N$  and  $N_f$  are log normally distributed, once more a difference distribution may be defined as:

$$D = \ln N_f - \ln N \quad (72)$$

$$E(D) = \bar{D} \cong \ln \bar{N}_f - \ln \bar{N} \quad (73)$$

$$\sigma^2(D) = \text{var}(D) = cv^2(N_f) + cv^2(N) \quad (74)$$

Reliability in this case is defined as the probability that  $D$  is greater than zero, that is:

$$R = \text{Prob} [d > 0] \quad (75)$$

As before, the design value of  $D_d$  is zero, which means that:

$$Z_R = \frac{D_d - \bar{D}}{\sigma_D} \quad (76)$$

$$Z_R = \frac{-\ln \bar{N}_f + \ln \bar{N}}{\sigma_D} \quad (77)$$

which gives:

$$\ln \bar{N}_f = \ln \bar{N} - Z_R \sigma_D \ln e \quad (78)$$

or

$$\bar{N}_f = \bar{N} e^{-Z_R \sigma_D} \quad (79)$$

The normal variable  $Z_R$  is chosen from the same table as before.

Whether  $N_f$  and  $N$  are distributed normally or log normally, the relationship between the expected value of traffic,  $\bar{N}$ , and that of the number of load applications to failure,  $\bar{N}_f$ , depends only upon the normal variable,  $Z_R$ , for the desired level of reliability and the coefficients of variation of  $N$  and  $N_f$ .

As noted previously, these coefficients of variation can be estimated realistically from the Taylor series expansion of the equations used to

predict  $N$  and  $N_f$ . The coefficient of variation of  $N_f$  depends upon the coefficient of variation of  $K_1$  and  $K_2$  and their correlation coefficient, all of which can be, and have been, measured extensively in the laboratory. The coefficient of variation of the number of traffic load applications,  $N$ , depends upon the coefficients of variation of the average daily traffic, the traffic growth rate, and the load equivalence factor for the traffic stream, all of which can be estimated realistically from existing traffic data records.

### Relation of Reliability to the Appearance of Cracking

Reliability can also be based upon the appearance of cracking and, in this case, reliability can be defined in the following equivalent ways:

$$R = \text{Prob} [c_{\max} > c]$$

$$R = \text{Prob} [c_{\max} - c > 0]$$

$$R = 1 - \int_{c_{\max}}^{\infty} \rho(x) dx$$

$$R = \int_{-\infty}^{c_{\max}} \rho(x) dx$$

where

$$\begin{aligned} c_{\max} &= \text{the maximum acceptable value of the area of cracking} \\ \rho(x) &= \text{the probability density function for cracking} \end{aligned}$$

The probability density functions for cracking have been discussed previously as being Gumbel, Weibull, or log normal. All of them make use of the formulation that the area of cracking is equal to the probability that the damage ratio,  $\frac{N}{N_f}$ , is greater than 1.0. It is recalled that the definition of the cracked area is:

$$c = \int_{D-1}^{D-\infty} \rho(x) dx \quad (81)$$

where

$$x = \frac{D - \bar{D}}{\sigma_D} \text{ (normal distribution)}$$

$$x = D \text{ (Gumbel and Weibull Distributions)}$$

$$D = \frac{N}{N_f}$$

The limits of integration of  $x$  are:

$$D = 1, x_1 = \frac{1 - \bar{D}}{\sigma_D} \text{ (normal distribution)} \quad (82)$$

when

$$D = 1, x_1 = 1 \text{ (Gumbel and Weibull)} \quad (83)$$

$$D = \infty, x = \infty \quad (84)$$

so that:

$$c = \int_{x_1}^{\infty} \rho(x) dx \quad (85)$$

An estimate of the variance of cracking may be obtained from this definition of cracking, since  $c$  is a function of both  $N$  and  $N_f$ ,

$$\text{var}(c) = \left[ \frac{\partial c}{\partial N} \right]^2 \text{var}(N) + \left[ \frac{\partial c}{\partial N_f} \right]^2 \text{var}(N_f) \quad (86)$$

where

$$\frac{\partial c}{\partial N} = \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial D} \cdot \frac{\partial D}{\partial N} = \frac{\partial c}{\partial x} \cdot \frac{1}{\sigma_D} \cdot \frac{1}{N_f} \quad (87)$$

$$\frac{\partial c}{\partial N_f} = \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial D} \cdot \frac{\partial D}{\partial N_f} = \frac{\partial c}{\partial x} \cdot \frac{1}{\sigma_D} \cdot \left[ -\frac{N}{N_f^2} \right] \quad (88)$$

Thus,

$$\text{var}(c) = \left( \frac{dc}{dx} \right)_{D=\bar{D}}^2 \cdot \left( \frac{\bar{N}}{\sigma_D \bar{N}_f} \right)^2 \cdot [cv^2(N) + cv^2(N_f)] \quad (89)$$

$$\text{var}(c) = \left( \frac{dc}{dx} \right)_{D=\bar{D}}^2 \cdot \frac{\bar{D}^2}{\sigma_D^2} \cdot [cv^2(N) + cv^2(N_f)] \quad (90)$$

But since:

$$cv^2(D) = \frac{\sigma_D^2}{\bar{D}^2} = cv^2(N) + cv^2(N_f) \quad (91)$$

Then:

$$\text{var}(c) = \left( \frac{dc}{dx} \right)_{D=\bar{D}}^2 \quad (92)$$

Because  $c$  is defined by the integral:

$$c = \int \rho(x) dx \quad (93)$$

its derivative is defined by the function being integrated.:

$$\frac{dc}{dx} = \frac{d}{dx} \left[ \int \rho(x) dx \right] = \rho(x) \quad (94)$$

$$\left( \frac{dc}{dx} \right)_{d=\bar{D}} = [\rho(x)]_{\substack{x=0 \text{ (Normal, log normal)} \\ x=\bar{D} \text{ (Gumbel and Weibull)}}} \quad (95)$$

And the variance of cracking,  $\text{var}(c)$ , is given by:

$$\text{var}(c) \cong [\rho^2(x)]_{\bar{x}} \substack{x=0 \text{ (Normal, log normal)} \\ \bar{x}=\bar{D} \text{ (Gumbel and Weibull)}} \quad (96)$$

For the three probability density functions appropriate for cracking, the variances of cracking are as follows:

Gumbel:

$$\rho(\bar{x}) = \beta \rho \bar{x}^{-(1+\beta)} e^{-\left(\frac{\rho}{\bar{x}}\right)} \quad (97)$$

$$\bar{x} = \bar{D}$$



Weibull:

$$\rho(\bar{x}) = \gamma \lambda (\lambda \bar{x})^{\gamma-1} e^{-(\lambda \bar{x})^\gamma} \quad (98)$$

$$\bar{x} = \bar{D}$$

Log Normal:

$$\rho(\bar{x}) = \frac{1}{\gamma \bar{\sigma} \sqrt{2\pi}} \cdot \exp \left[ -\frac{1}{2} \left( \frac{\bar{x}}{\gamma} \right)^2 \right] \quad (99)$$

where, in the this case:

$$\bar{x} = 0$$

and

$$\sigma(\ln D) = \exp \left[ \bar{D} + \frac{\gamma^2}{2} \right] [\exp(\gamma^2) - 1]^{\frac{1}{2}} \quad (100)$$

and

$$\gamma^2 = \text{var}(D) = \bar{D}^2 \text{cov}^2(D) \quad (101)$$

The use of this estimated variance of cracking is the same as before. The reliability of the pavement is defined as the probability that the actual cracked area will not exceed the maximum allowable  $c_{\max}$ , when the damage ratio,  $\frac{\bar{N}}{\bar{N}_f}$ , equals 1.0. This is illustrated in Figure 7:

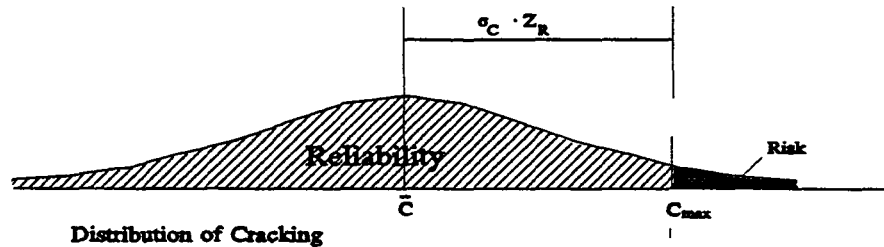


Figure 7. Pavement Reliability with Reference to Pavement Cracking.

$$c_{\max} = \int_{\substack{D = D - 1 \\ N = N_f}}^{\infty} \rho(x) dx \quad (102)$$

$$\bar{c} = \int_{\bar{x}}^{\infty} \rho(x) dx \quad (103)$$

$$c_{\max} - \bar{c} = Z_R \cdot [\text{var}(c)]^{1/2} \quad (104)$$

$$\bar{x} = \frac{1 - \bar{D}}{\sigma_D} \quad (\text{normal distribution}) \quad (105)$$

$$\bar{x} = \bar{D} \quad (\text{Gumbel and Weibull distribution}) \quad (106)$$

$$\bar{x} = -\frac{\ln(\bar{D})}{\sigma (\ln D)} \quad (\text{log normal distribution}) \quad (107)$$

The reliability multiplier,  $Z_R$ , must be taken from the same probability density function which is used in the definition of cracking. In the case of the log normal distribution of traffic,  $N$ , and load applications to failure,  $N_f$ , the cracking area itself is normally distributed and the  $Z_R$  values are taken from the normal variable tables given previously. The pavement must be designed so that the expected traffic ( $\bar{N}$ ) will cause the amount of damage  $\bar{c}$ , given by the equation below:

$$c_{\max} = \bar{c} + Z_R \cdot [\text{var}(c)]^{1/2} \quad (108)$$

Put in another way, the amount of cracking to be expected after the application of  $\bar{N}$  loads should be no more than

$$\bar{c} = c_{\max} - Z_R [\text{var}(c)]^{1/2} \quad (109)$$

### Relation of Reliability to Construction Specifications

The relation of reliability to construction specifications can be illustrated using a simplified form of the number of load cycles to failure,

$N_f$ . It is recalled that cracking damage,  $D$ , is defined as a ratio of traffic,  $N$ , to the load applications to reach failure,  $N_f$ , and an appropriate probability density function (e.g., log normal, Gumbel, or Weibull). The traffic,  $N$ , is predicted as:

$$N(T) = l \left[ a_v T + r \frac{T^2}{2} \right] \quad (110)$$

where

- $a_v$  = the initial number of vehicles per day
- $T$  = the elapsed time, in days
- $r$  = the rate of increase of traffic per day
- $l$  = the load equivalent factor for the traffic stream.

Each of the variables,  $a_v$ ,  $r$ , and  $l$  have expected values and variances. Thus, for any given time,  $t$ , the traffic,  $N$ , will have an expected value and variance.

The number of load cycles to failure is also a function of several variables as simplified below for the purposes of this illustration:

$$N_f = c \frac{d^{1-\frac{n}{2}} \sigma_t^2 s}{E^{\frac{n}{2}}} \left[ \frac{1}{\epsilon} \right]^n \quad (111)$$

where

- $d$  = the thickness of the surface layer
- $\sigma_t$  = the tensile strength of the surface layer
- $E$  = the modulus of the surface layer
- $n$  = the fracture exponent of the surface layer
- $\epsilon$  = the strain at the bottom of the surface layer
- $s$  = speed of travel
- $c$  = a constant of proportionality

The variables  $d$ ,  $\sigma_t$ ,  $E$ , and  $n$  are controlled by the quality of construction and their expected values and coefficients of variation in the field dictate the expected value and variance of  $N_f$ .

The reliability of a pavement depends upon the expected value and

variance of the damage ratio, D, which is given

by:

$$\bar{D} = E(D) = E\left[\frac{N}{N_f}\right] \cong \left[\frac{\bar{N}}{N_f}\right] [1 + cv^2(N_f)] \quad (112)$$

$$\sigma_D^2 = \text{var}(D) = \text{var}\left[\frac{N}{N_f}\right] \cong \left[\frac{\bar{N}}{N_f}\right]^2 [cv^2(N) + cv^2(N_f)] \quad (113)$$

From this result, it is seen that the expected values and variances of N and  $N_f$  uniquely establish both the expected value and variance of the damage and consequently, the level of reliability of the pavement. To determine how construction specifications affect reliability, it is necessary to go a step farther to find how both the N and  $N_f$  expected values and variances depend upon the values previously noted:

For the traffic, N

$a_v$   
r  
 $\ell$

For the load applications to failure,  $N_f$

d  
 $\sigma_t$   
E  
n  
 $\epsilon$

The expected value of the traffic ( $E(N)$ ) is:

$$\bar{N} = E[N] \cong \bar{\ell} \left[ \bar{a}_v T + \bar{r} \frac{T^2}{2} \right] + T \text{cov}(a_v, \ell) + \frac{T^2}{2} \text{cov}(\ell, r) \quad (114)$$

The variance of the traffic (Var (N)) is:

$$\begin{aligned}
 \sigma^2(N) - \text{var}(N) \cong & \left[ \bar{a}_v T + \bar{r} \frac{T^2}{2} \right]^2 \text{var}(\ell) + (\bar{\ell} T)^2 \text{var}(a_v) \\
 & + \left[ \frac{\bar{\ell} T^2}{2} \right]^2 \text{var}(r) \\
 & + 2 \left[ \bar{a}_v T + \bar{r} \frac{T^2}{2} \right] (\bar{\ell} T) \text{cov}(a_v, \ell) \\
 & + 2 \left[ \bar{a}_v T + \bar{r} \frac{T^2}{2} \right] \left[ \frac{\bar{\ell} T^2}{2} \right] \text{cov}(\ell, r) \\
 & + 2 (\bar{\ell} T) \left[ \frac{\bar{\ell} T^2}{2} \right] \text{cov}(a_v, r)
 \end{aligned} \tag{115}$$

The covariance terms are included because there is a possible correlation between initial traffic rate,  $a_v$ ; rate of increase,  $r$ ; and load equivalence factor per vehicle,  $\ell$ . The covariance terms may also be written as

$$\begin{aligned}
 \text{cov}(a_v, \ell) &= \rho_{a,\ell} [\text{var}(a_v) \text{var}(\ell)]^{1/2} \\
 \text{cov}(\ell, r) &= \rho_{\ell,r} [\text{var}(\ell) \text{var}(r)]^{1/2} \\
 \text{cov}(r, a_v) &= \rho_{r,a} [\text{var}(r) \text{var}(a_v)]^{1/2}
 \end{aligned}$$

and the coefficients  $\rho_{a,\ell}$ ,  $\rho_{\ell,r}$ , and  $\rho_{r,a}$  are the correlation coefficients for the noted pairs of variables. The expected value of the traffic to reach failure is:

$$\bar{N}_f = E(N_f) = c \bar{d} \bar{\sigma}_t^2 \bar{s} \left[ \frac{1}{(\bar{d} \bar{E})^{1/2} \bar{\epsilon}} \right]^{\bar{n}} + \text{other terms involving coefficients of}$$

variation of each of the variables.

(119)

The expanded form of this expression is:

$$\begin{aligned}
E(N_f) = \hat{N}_f & \left[ 1 - \frac{\bar{d}^2 \bar{n}}{2} \left[ 1 - \frac{\bar{n}}{2} \right] cv^2(d) \right. \\
& + \frac{1}{2} cv^2(\sigma_t) - \frac{\bar{n}}{4} cv^2(E) \\
& + \frac{s}{2} cv^2(s) + \frac{\bar{n}^2}{2} \left\{ \ln \left[ \frac{1}{(\bar{E} \bar{d})^{1/2} \epsilon} \right] \right\}^2 cv^2(n) \\
& \left. - \bar{n} \rho_{\sigma E} cv(E) cv(\sigma_t) \right] \quad (120)
\end{aligned}$$

$$\begin{aligned}
\frac{var(N_f)}{\hat{N}_f^2} \cong cv^2(N_f) & = \left[ 1 - \frac{\bar{n}}{2} \right]^2 cv^2(d) + \frac{1}{4} cv^2(\sigma_t) \\
& + \frac{\bar{n}^2}{4} cv^2(E) + cv^2(s) \\
& + \bar{n}^2 \left\{ \ln \left[ \frac{1}{(\bar{E} \bar{d})^{1/2} \epsilon} \right] \right\}^2 cv^2(n) \\
& - \frac{\bar{n}}{2} \rho_{\sigma E} \cdot cv(\sigma_t) cv(E) \quad (121)
\end{aligned}$$

The term  $\hat{N}_f$  is an approximation of  $E(N_f)$  and is equal to

$$\hat{N}_f = c \bar{d} \bar{\sigma}_t^2 \bar{s} \left[ \frac{1}{(\bar{E} \bar{d})^{1/2} \epsilon} \right]^{\bar{n}} \quad (122)$$

In the formulation given above, the strain ( $\epsilon$ ) has been assumed to be deterministic and only the variables  $\sigma_t$  and  $E$  have been assumed to be correlated. The correlation coefficient between the two is  $\rho_{\sigma E}$ . The values  $\bar{d}$ ,  $\bar{E}$ ,  $\bar{\sigma}_t$ ,  $\bar{s}$ , and  $\bar{n}$  are the expected values of those individual variables.

As can be seen in this formulation, the only values of these variables that are used in determining the reliability of a pavement are the expected values and variances. Construction specifications are written explicitly to control the layer thickness,  $d$ , and various material properties are controlled directly by imposing specification limits on tensile strength (concrete pavement) or indirectly by setting specification limits on compacted density and asphalt content (asphalt pavement).

If one of these material properties is used in specifications to control quality, it is readily apparent from the foregoing that it cannot be considered to control the reliability of the pavement design. Furthermore, it is readily observed that:

1. In the coefficient of variation of traffic ( $cv(N)$ ) the load equivalence factor ( $\ell$ ) per vehicle appears in all of the terms.
2. In the coefficient of variation of the number of load applications to failure ( $cv(N_f)$ ), the fatigue exponent,  $n$ , appears in almost all of the terms.
3. The coefficients of variation of  $N$  and  $N_f$  can be expressed as weighted sums of the coefficients of variation of several other variables upon which they depend explicitly. All of these  $cv$  values can normally be assumed accurately enough for the purposes of design reliability by a person with sound experience.
4. Controlling the construction quality by monitoring test values of a single variable does not, in itself, control the design reliability of the pavement.

The relation between the minimum tensile strength of a paving surface material ( $s_m$ ) and the values needed for design reliability,  $\bar{\sigma}_t$  and  $cv(\sigma_t)$ , is one which requires the designer to assume a coefficient of variation and the probability density function of the variable selected for control. What is specified is a minimum strength ( $s_m$ ) and a percentage of the tests that may fail to achieve that strength ( $\rho$ ).

If the tests are assumed to follow a normal distribution pattern, the normal variate,  $Z_c$ , for the construction quality control test may be found from the minimum percentage,  $\rho$ , by:

$$\rho = \text{erf}(Z_c) \quad (123)$$

or

$$\rho = \int_{-\infty}^{Z_c} e^{-\frac{x^2}{2}} dx \quad (124)$$

This is illustrated in Figure 8 with:

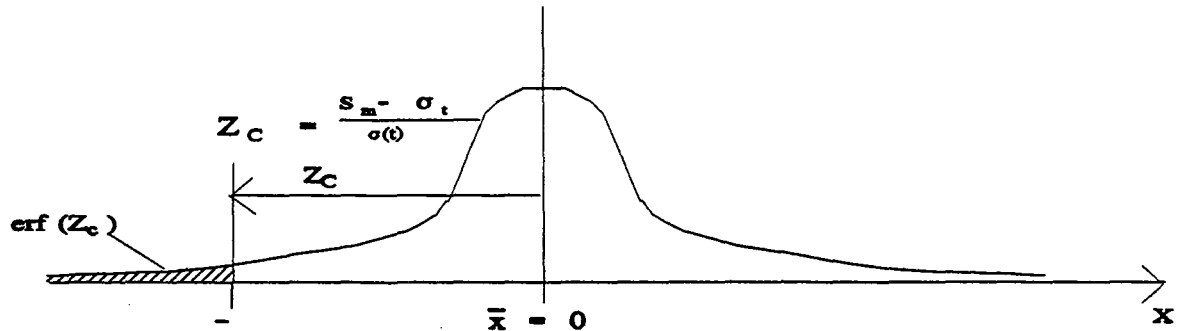


Figure 8. Construction Quality Control Distribution.

$$x = \frac{s - \bar{\sigma}_t}{\sigma(t)} \quad (125)$$

where

$\frac{s}{\bar{\sigma}_t}$  = strength  
 $\bar{\sigma}_t$  = mean strength  
 $\sigma(t)$  = the standard deviation of strength.

Once the value of the normal variate,  $Z_c$ , is known, the mean value of the tensile strength,  $\bar{\sigma}_t$ , may be found only if the coefficient of variation of the strength is also assumed to be known. The mean strength is:

$$\bar{\sigma}_t = \frac{s_m}{1 - Z_c \text{ cv}(\sigma_t)} \quad (126)$$

Setting a ratio of the minimum strength to the mean strength along with setting a minimum percent of tests failing,  $\rho$ , is the same as setting a



maximum allowable coefficient of variation. As before,  $\rho$  determines  $Z_c$  and the strength ratio ( $s_m/\bar{\sigma}_t$ ), together with  $Z_c$ , specifies the coefficient of variation:

$$cv(\sigma_t) = \frac{1}{Z_c} \left[ 1 - \frac{s_m}{\sigma_t} \right] \quad (127)$$

The same reasoning applies to construction specifications controlling the thickness and percent allowable variation of pavement layer thicknesses.

The logic is inescapable: setting specifications is either arbitrary (and therefore irrational) or it is based upon knowledge and experience. If this latter is the case, then regardless of the specification method that is used,

1. setting a minimum and a maximum percent to fall below it, or
2. setting a ratio of the minimum to the mean and a maximum percentage to fall below it,

the knowledge and experience upon which it is based is upon a known coefficient of variation and a known, achievable, mean value of the specified variable. Stated in another way: to imply that specifications can be set without an explicit or at least intuitive knowledge based upon experience of the mean and coefficient of variation of the specified variable, is absurd.

There are other important lessons to be learned from the expressions given above for the expected value and coefficient of variation of  $N_f$  and the number of load applications to reach failure. Both of the expressions for  $E(N_f)$  and  $cv(N_f)$  include the effect of the covariance of the modulus and the tensile strength, variables which are known to be positively correlated in both asphalt and concrete. The lessons are as follows:

1. the correlation coefficient between the two is positive, i.e.,  
 $\rho_{\sigma E} = (+)$ .
2. both  $E(N_f)$  and  $cv(N_f)$  are related to the covariance (an inherently positive quantity) of  $E$  and  $\sigma_f$  by
  - a. a negative sign
  - b. the fatigue exponent,  $n$ , as a weighing multiplier.

This means that the covariance of these two variables (or any two positively correlated variables) reduces both  $E(N_f)$  and  $cv(N_f)$ , and the size of the fatigue exponent governs how much that reduction is. It is a strange, but true, fact that because the tensile strength and modulus of a material are positively correlated, the expected traffic that can be carried by a pavement is reduced. If a material is used in which stiffness and strength are negatively correlated, then it will increase the expected life of a pavement. Stress-strain curves for the two types of material are illustrated below.

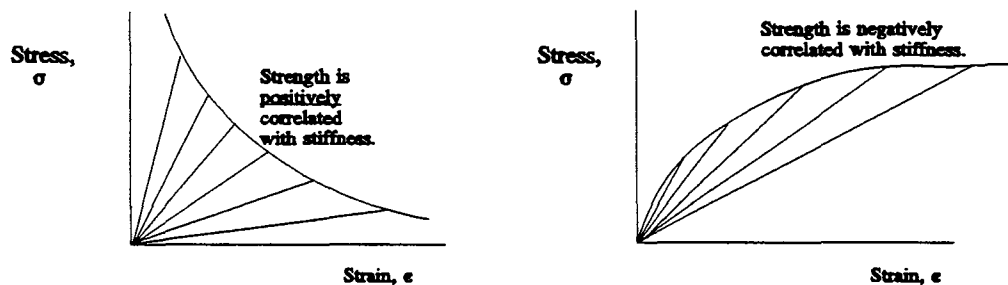


Figure 9. Correlation Between Material Strength and Stiffness.

Thus, the covariance of material properties has a direct effect upon the life and reliability of the pavement, with a positive correlation having as decremental effect and negative correlation having an incremental effect.

Specification limits should not be set without recognizing this important effect. This can be done by setting upper limits on the positive correlation coefficient and no lower limits on negative correlation coefficients.

#### APPEARANCE OF DISTRESS ON REHABILITATED PAVEMENT - RELIABILITY OF OVERLAYS

When a pavement is overlaid, there are some areas in the old pavement surface that are weaker than others. The percent of the total area of the old pavement that is thus weakened is  $c$ , (Figure 10), the percent area of cracking at the time of rehabilitation. The damage to the overlay above these portions of the pavement will occur more rapidly than elsewhere, meaning that the number of load cycles to reach failure will be smaller here than elsewhere.

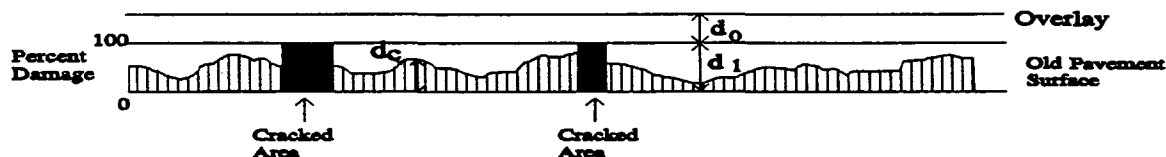


Figure 10. Variation in Pavement Condition at the Time of Overlay.

At any given point along the pavement length, the crack depth will have reached to some proportion of the old surface layer thickness. If the crack depth is  $d_c$  and the depth of the old pavement surface layer is  $d_1$ , the relation between the two is:

$$d_c = d_1 p(x) \quad (128)$$

In these areas, cracks must grow through a distance of:

$$= d_0 + [1 - p(x)] d_1 \quad (129)$$

where

$d_0$  = the thickness of the overlay

In the cracked areas, the cracks must grow upward through the overlay thickness alone. Making use of some of the principles of fracture mechanics, it is known that the number of load cycles to reach failure in the cracked areas  $N_{fc}$ , is proportional to:

$$N_{fc} \propto \frac{d_0^{1 - \frac{n}{2}} \sigma_{t0}^2 s}{E_0^{\frac{n}{2}}} \quad (130)$$

where

$\sigma_{t0}$  = the tensile strength of the overlay  
 $s$  = the speed of travel  
 $E_0$  = the elastic modulus of the overlay material  
 $n$  = the fracture exponent of the overlay material

In the areas where the cracks had not penetrated through the old surface layer, the number of load applications to reach failure,  $N_{fu}$ , is proportional to:

$$N_{fu} \propto \left[ (d_0 + d_1)^{1 - \frac{n}{2}} - \frac{d_1^{1 - \frac{n}{2}}}{E_0^{\frac{n}{2}}} \cdot \sigma_{t0}^2 s \right] + \left[ \frac{d_1^{1 - \frac{n}{2}}}{E_1^{\frac{n}{2}}} \sigma_{t1}^2 s \right] [1 - p(x)^{1 - \frac{n}{2}}] \quad (131)$$

where

$E_1$  = the elastic modulus of the old surface layer  
 $\sigma_{t1}$  = the tensile strength of the old surface layer  
 $n_1$  = the fracture exponent of the old surface material

Typical values of the fracture exponents are: asphalt concrete, 2-4; portland cement concrete, 12-16.

The ratio of the two numbers of load cycles to reach failure is:

$$\frac{N_{fc}}{N_{fu}} = \frac{1}{\left[1 + \frac{d_1}{d_0}\right]^{1 - \frac{n}{2}} - \left[\frac{d_1}{d_0}\right]^{1 - \frac{n}{2}} \left\{1 - \left[\frac{\sigma_{t_1}}{\sigma_{t_0}}\right]^2 \left[\frac{E_0}{E_1}\right]^{\frac{n}{2}} [1 - p(x)]^{1 - \frac{n}{2}}\right\}} \quad (132)$$

The above expression assumes that the fracture exponent (n) is the same in the overlay and in the old surface layer. If, in addition to this, it is also assumed that:

$$\sigma_{t_1} = \sigma_{t_0} \quad (133a)$$

$$E_1 = E_0 \quad (133b)$$

$$\bar{p}(x) \cong \frac{1 - c_r}{2} \quad (133c)$$

where the average value of  $p(x)$  is  $\bar{p}(x)$  and  $c_r$  is the area of cracking at the time of rehabilitation. Under these assumptions, the ratio of the number of load cycles to failure is:

$$\frac{N_{fc}}{N_{fu}} = \frac{1}{\left[1 + \frac{d_1}{d_0}\right]^{1 - \frac{n}{2}} - \left[\frac{d_1}{d_0} \cdot \frac{1 - c_r}{2}\right]^{1 - \frac{n}{2}}} = f_u \quad (134)$$

This expression leads to the definition of damage in the "uncracked" area  $(1 - c_r)$  and in the "cracked" area  $(= c_r)$ .

Damage in the "cracked" zone:

$$D_c = \frac{N}{N_{fc}} \quad (135)$$

Damage in the "uncracked" zone:

$$D_u = \frac{N}{N_{fu}} = \frac{N}{N_{fc}} \cdot \frac{N_{fc}}{N_{fu}} \quad (136)$$

$$D_u = \frac{N}{N_{fc}} \cdot \left[ \frac{1}{\left[ \left( 1 + \frac{d_1}{d_0} \right)^{1 - \frac{n}{2}} - \left( \frac{d_1}{d_0} \cdot \frac{1 - c_r}{2} \right)^{1 - \frac{n}{2}} \right]} \right] \quad (137)$$

$$D_u = \frac{N}{N_{fc}} \cdot f_u \quad (138)$$

The total amount of cracking that occurs in the surface of the overlay is:

$$c = \int_{x_{c1}}^{\infty} \rho(x_c) dx_c + \int_{x_{u1}}^{\infty} \rho(x_u) dx_u \quad (139)$$

$$x_{c1} = \frac{1 - \bar{D}_c}{\sigma_{D_c}} \quad (\text{normal distribution}) \quad (140)$$

$$x_{u1} = \frac{1 - \bar{D}_u}{\sigma_{D_u}} = \frac{1 - \bar{D}_c f_u}{f_u \sigma_{D_c}} \quad (\text{normal}) \quad (141)$$

$$x_{c1} = \frac{-\ln \bar{D}_c}{\sigma(\ln D_c)} \quad (\text{Log normal distribution}) \quad (142)$$

$$x_{u1} = \frac{-\ln \bar{D}_u}{\sigma(\ln D_u)} = \frac{-\ln(f_u \bar{D}_c)}{\sigma[\ln(f_u D_c)]} \quad (\text{Log normal}) \quad (143)$$

$$x_{c1} = 1 \quad (\text{Gumbel and Weibull}) \quad (144)$$

$$x_{u1} = f_u x_{c1} = f_u \quad (\text{Gumbel and Weibull}) \quad (145)$$

The values of  $x_c$  and  $x_u$  have similar definitions:

Normal Distribution:

$$x_c = \frac{D_c - \bar{D}_c}{\sigma_{Dc}} \quad (146)$$

$$x_u = \frac{D_u - \bar{D}_u}{\sigma_{Du}} = \frac{D_c f_u - \bar{D}_c f_u}{f_u \sigma_{Dc}} \quad (147)$$

$$x_u = x_c \quad (148)$$

Log Normal Distribution:

$$x_c = \frac{\ln D_c - \ln \bar{D}_c}{\sigma(\ln D_c)} \quad (149)$$

$$x_u = \frac{\ln D_u - \ln \bar{D}_u}{\sigma(\ln D_u)} = \frac{\ln (f_u D_c) - \ln (f_u \bar{D}_c)}{\sigma[\ln (f_u D_c)]} \quad (150)$$

$$x_u = \frac{\ln D_c - \ln \bar{D}_c}{\sigma(\ln D_c)} \quad (151)$$

$$x_u = x_c \quad (152)$$

Gumbel and Weibull Distributions:

$$x_c = D_c \quad (153)$$

$$x_u = D_u = f_u D_c \quad (154)$$

$$x_u = f_u x_c \quad (155)$$

The variances of damage for the "cracked" and "uncracked" areas are as follows:

$$\sigma_{Dc} = \bar{D}_c \left[ cv^2(N) + cv^2(N_{fc}) \right]^{1/2} \quad (156)$$

$$\sigma_{Du} = f_u \sigma_{Dc} \quad (157)$$

The variances of cracking for the "cracked" and "uncracked" areas are as follows:

$$\text{var}(c) = \left[ \frac{dc}{dx} \right]_{D = \bar{D}_c}^2 + \left[ \frac{dc}{dx} \right]_{D = \bar{D}_c = f_u \bar{D}_c}^2 \quad (158)$$

$$\text{var}(c) = \left[ \rho^2(x_c) \right]_{x_c = \bar{x}_c} + \left[ \rho^2(f_u x_c) \right]_{x_c = \bar{x}_c} \quad (159)$$

where

$$\bar{x}_u = f_u \bar{x}_c \quad (160)$$

and

$$\bar{x}_c = \bar{x}_u = 0 \quad (\text{normal and Log normal}) \quad (161)$$

$$\bar{x}_c = \bar{D}_c \quad (\text{Gumbel and Weibull}) \quad (162)$$

$$\bar{x}_u = f_u \bar{D}_c \quad (\text{Gumbel and Weibull}) \quad (163)$$

Reliability is introduced into the design of an overlay by designing it to carry the design traffic when it reaches a design level of cracking,  $c_d$ , given by:

$$c_d = c_{\max} - Z_R [\text{var}(c)]^{1/2} \quad (164)$$

and the reliability factor,  $Z_R$ , is calculated using the appropriate probability density function.

## SUMMARY AND CONCLUSIONS

The correct application of reliability to pavement design is essential to the objectives of pavement design which are to produce quality pavements to serve the traveling public in comfort and safety, being built to be durable in service at a minimum life cycle cost. Reliability is a technical term being defined in mathematical terms and is therefore objective in its application. As applied to pavements, it makes use of either empirical or mechanistic pavement performance equations to predict an expected value and



variance of either the traffic or the distress for which the pavement must be designed. Explicit expressions are found for the quantities in terms of the expected values and coefficients of variation (cv's) of the independent variables which appear in the performance equation(s). Although ample data are available to determine these for all pavement independent variables, experienced engineers can estimate realistic values of the cv's almost as well as they can be measured.

Several examples of the application of reliability to pavement performance models are given in this paper including:

1. Proper methods of analyzing field data to develop an empirical equation for the number of load cycles to reach pavement failure due cracking.
2. Pavement performance equations for cracking which incorporate reliability in determining the number of load cycles for which the pavement should be designed.
3. Relations of reliability to construction specifications.
4. Methods of incorporating reliability into the design of overlays while taking into account the variability of the cracking that occurred in the old pavement.

In all cases, the mathematical nature of reliability was demonstrated. It is because of its ability to be defined in precise, unambiguous terms that makes it simple to apply and possible to set objective standards for the performance of pavements while taking into account the variability of the factors which control the performance. Reliability must be applied correctly in designing pavements in order to achieve its objective of providing an acceptable level of risk that pavements will carry the traveling public in comfort and safety while being built durably at minimum life cycle costs.

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## APPENDIX I

### STATISTICAL VARIABLES AND FUNCTIONS

Since in depth discussions of reliability involve use of some statistical terminology, it is incumbent to provide definitions of commonly referred to reliability terms.

A list of terminology is provided as:

1. variables
2. frequency distribution
3. populations and samples
4. probability density function
5. expected value
6. variance
7. standard deviation
8. coefficient of variation
9. covariance
10. cumulative probability function

Variables. The term variable has become practically all-inclusive in statistics. It refers to something being observed that exhibits variation. The variation may be in part due to errors of measure, the natural characteristics of certain material properties, or planned variation that is imposed externally. Some variables are controlled as part of a planned experiment. Other variables which result from experimentation vary from repeated trials of the same combinations of initial conditions. In other words, yield variables display a certain randomness in their behavior. Dividing the variables associated with any system into those which can be controlled and act upon the system (i.e. independent variables) is the basis of understanding of the application of statistics in design reliability.

Frequency distribution. The distribution of a random variable refers to a profile of the variable that contains all the relevant information about the statistical properties of that random variable [7]. In many

instances, it is unnecessary to use all the information contained in the distribution, but rather a few properties of the distribution may be sufficient. Two of the most frequently useful pieces of information are the location of the distribution and its dispersion.

Populations and samples. Statistically speaking, it is important to distinguish between populations and samples. The population (sometimes called the universe) comprises all of the possible observations that exist. A sample is a collection of observations actually taken. Completely enumerated, the sample and the population become one in the same. Normally, the sample is a subset of the total population. Statistical variables apply to both samples and the population. However, statistical variables of a sample only permit inference of certain information about the population. Without complete enumeration, sample statistics cannot be applied with certainty to the population statistics.

The variables that are used in determining reliability which are defined mathematically are delineated and follow below:

Probability Density Function (PDF). A probability density function is any function which has, beneath the curve, a total area equal to 1.0. It does not have to be symmetric, or to have one peak, or to have any of the characteristics of the commonly recognized normal distribution curve. Mathematically, the probability density function,  $f(x)$ , is

$$1 = \int_{\text{lower limit}}^{\text{upper limit}} f(x) dx$$

The lower limit may be  $-\infty$  or 0, or any other. The upper limit may be 0 or  $+\infty$  or any other known upper limit.

Quantitative descriptors of a random variable related to the performance of a pavement system may represent a range of events of value

of the performance variable. Since the value of the performance variable is a random event, it can assume a numerical value only with an associated probability [5]. The description or measure of the probability associated with all the values of a random variable is the probability distribution.

The form of the distribution function, equivalently referred to as probability density function (PDF), describes completely the probabilistic characteristics of a random variable. These functions are discussed later in greater detail but approximate descriptors of a random variable are frequently employed in the form of main descriptors. Main descriptors refer to the central value of the random variable and a measure of the dispersion associated with the random variable values.

In mechanistic design approaches, the assumed distribution function may be verified based on data collected from field studies of pavement performance. However, if the type of data is not available, then it may be appropriate to assume a distribution function (i.e. normal distribution) until such data is available for verification purposes. From a practical perspective, a defined distribution in addition to the principal quantities will provide the information to evaluate the properties of the random variable.

Expected Value. The central value of a random variable can be referred to as the mean or expected value. This type of measure (such as the average) is of natural interest since there is a range of possible values associated with a random variable. Therefore, as this range of values are distributed and the different values of the range variables within the distribution are associated with different probabilities. With the noted distribution, a weighted average can be determined referred to as the mean

value or the expected value of this random variable [5]. Another symbol commonly used for the expected value is  $\mu$ , which stands for the mean.

If a random variable ( $x$ ) is continuous over its range ( $R$ ) of values, then mean value may be found from the PDF ( $f_x(x)$ ) as:

$$E(x) = \int_R x f_x(x) dx \quad (1)$$

Also applicable to mechanistic design concepts is a generalization of equation for a function of  $x$ . Given a function  $g(x)$ , its weighted average or mathematical expectation is found as:

$$E(g(x)) = \int_R g(x) f_x(x) dx \quad (2)$$

Other statistics related to the mean are the median and the mode. The median is a type of statistical average in which 50 percent of the observed values are greater and 50 percent are smaller. The median is usually more appropriate to use than the mean with highly skewed distributions. The mode is the value in a distribution that occurs with the greatest frequency. The mean, median, and mode are illustrated in Figure 1. Another useful statistical term is the range. It is defined as the difference between the layout and the smallest observations in a sample distribution.

Variance. A measure of the dispersion of a random is noted as the variance. The standard deviation ( $\sigma$ ), which is the square root of the variance, can also serve to indicate the degree of dispersion for a given distribution. The variable distribution relates the level or variability which is the quantity that indicates how closely the values of the random variable occur around the weighted average or mean. The variability is dependent upon the derivations from the central value. The dependency should not be a function of whether a deviation is above or below the central value.

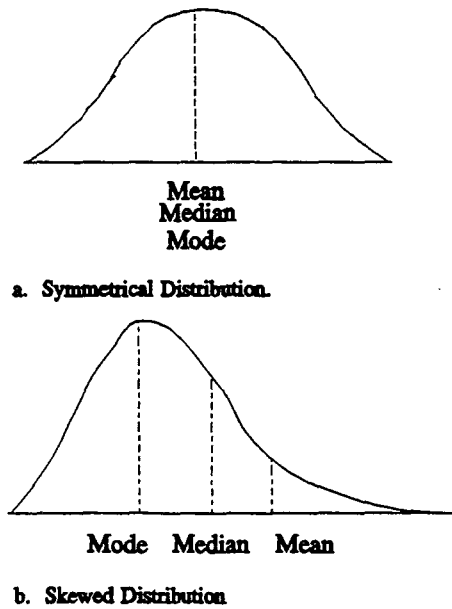


Figure 1. Frequency Distribution Illustrating the Relationship Between the Mean, the Median, and the Mode.

The first moment about the expected value is always zero, no matter what the probability density function.

$$\begin{aligned} E[(x-\mu)] &= \int_{-\infty}^{\infty} (x-\mu) f_x(x) dx \\ &= 0 \end{aligned}$$

If the deviations about the central value are found, a measure of dispersion is obtained which is the variance ( $\text{Var}(x)$ ):

$$\begin{aligned} \text{Var}(X) &= \int_R (x-\mu_x)^2 f_x(x) dx \\ &= \int_R (x^2 - 2\mu_x x + \mu_x^2) f_x(x) dx \\ &= E(X^2) - 2\mu_x E(x) + \mu_x^2 \\ &= E(X^2) - \mu_x^2 \end{aligned} \tag{3}$$



The variance can also be found from the second moment about the mean which is the expected value of  $(x-\mu)^2$ . The term  $E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x) dx$  is known as the mean-square value of  $x$  [5]. The standard deviation ( $\sigma$ ), previously noted as the square root of the variance can be determined as:

$$\sigma_x = \sqrt{\text{Var}(X)} \quad (4)$$

A measure of whether the dispersion is large or small, (as given by the variance or the standard deviation) the coefficient of variation (cv) is used:

$$cv = \frac{\sigma_x}{\mu_x} \quad (5)$$

which is a measure of dispersion relative to the central value.

Covariance. If two variables,  $x$  and  $y$ , are multiplied together to form the product,  $xy$ , the expected value of the product is

$$\begin{aligned} E[(x,y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{x,y}(x,y) dx dy \\ &= E(x) E(y) + \rho \sigma(x) \sigma(y) \end{aligned} \quad (6)$$

where

$f_{x,y}(x,y)$  = the joint probability density function of  $x$  and  $y$

$\rho$  = correlation coefficient between  $x$  and  $y$

$\sigma(x)$  = the standard deviation of  $x$

$\sigma(y)$  = the standard deviation of  $y$

The last term in the expected value of the product of  $x$  and  $y$  is called the covariance of  $x$  and  $y$ .

$$\text{cov}(x,y) = \rho \sigma(x) \sigma(y)$$

It is equal to zero only if the correlation coefficient,  $\rho$ , is zero. In

general the expected value of a function of  $x$  and  $y$ ,  $g(x,y)$ , is given by

$$E[g(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{x,y}(x,y) dx dy \quad (7)$$

The coefficient of variation of a single variable  $cv(x)$  should not be confused with the covariance of two variables which is written as  $cov(x,y)$ .

### Functions Used in Reliability

Many functions which are used in the determination of reliability cannot be integrated in closed form. Instead, use is made of a Taylor's series approximation (assuming a normal distribution).

$$E[g(x)] = \int_{-\infty}^{\infty} f_x(x) g(x) dx$$

Because  $g(x)$  may be represented as a Taylor Series in the form:

$$g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2!}(x-\mu)^2 + \dots \quad (8)$$

The expected value of the function may be expressed in a similar form:

$$\begin{aligned} E[g(x)] &= \int_{-\infty}^{\infty} g(\mu) f_x(x) dx + \int_{-\infty}^{\infty} g'(\mu)(x-\mu) f_x(x) dx \\ &+ \int_{-\infty}^{\infty} \frac{g''(\mu)}{2} (x-\mu)^2 f_x(x) dx + \dots \end{aligned} \quad (9)$$

$$E[g(x)] = g(\mu) + \frac{g''(\mu)}{2} \text{Var}(x) + \dots$$

The variance of a function is approached in the same way, beginning with the relation:

$$\text{Var}[g(x)] = E[g^2(x)] - E^2[g(x)] \quad (10)$$

and arriving at the approximation using the Taylor series expansion of:

$$\begin{aligned} \text{Var}[g(x)] &= [g'(\mu)]^2 \text{Var}(x) \\ &+ \frac{1}{4} [g''(\mu)]^2 \text{Var}^2(x) [\beta_2 - 1] \\ &+ \beta_1 g'(\mu) g''(\mu) \text{Var}^{\frac{3}{2}}(x) + \dots \end{aligned} \quad (11)$$

where

$B_1$  = a measure of the skewness of  $f_x(x)$

$B_2$  = a measure of the kurtosis of  $f_x(x)$

when  $f_x(x)$  is symmetrical,  $B_1 = 0$ ; when it is normal,  $B_2 = 3$ .

The expected value of a function of two variables is arrived at using a two-dimensional Taylor series to give:

$$\begin{aligned} E[g(x,y)] &= g(\mu_x, \mu_y) + \frac{1}{2} g_{xx}(\mu_x, \mu_y) \text{Var}(x) \\ &+ \frac{1}{2} g_{yy}(\mu_x, \mu_y) \text{Var}(y) \\ &+ g_{xy}(\mu_x, \mu_y) \text{cov}(x,y) + \dots \end{aligned} \quad (12)$$

The variance of a function of two variables is given by:

$$\begin{aligned} \text{Var}[g(x,y)] &= [g_x(\mu_x, \mu_y)]^2 \text{Var}(x) \\ &+ [g_y(\mu_x, \mu_y)]^2 \text{Var}(y) \\ &+ 2 g_x(\mu_x, \mu_y) g_y(\mu_x, \mu_y) \text{cov}(x,y) + \dots \end{aligned} \quad (13)$$

In the equations given above,

$$g_{xx} = \frac{\partial^2 g}{\partial x^2}$$

$$g_{yy} = \frac{\partial^2 g}{\partial y^2}$$

$$g_x = \frac{\partial g}{\partial x}$$

$$g_y = \frac{\partial g}{\partial y}$$

$$g_{xy} = \frac{\partial^2 g}{\partial x \partial y}$$

$\mu_x$  = the expected value of  $x$

$\mu_y$  = the expected value of  $y$

## APPENDIX II

## PARAMETER DISTRIBUTION AND ESTIMATION

## Distribution of Variables

The numerical distribution of a random variable is circumscribed within the probability measure of the associated variable event or value. The distribution is defined by the probability density function (PDF) and can provide the basis for the probability measure corresponding to all values of a random variable. The probability measure of a random variable can also be described in terms of a probability distribution. A probability distribution is presented in the form of its cumulative distribution function (CDF) [5]:

$$F_X(x) \equiv P(X \leq x) \text{ for all } x$$

where

$F_X$  = cumulative distribution function  
 $X$  = random variable  
 $x$  = value of random variable

Any CDF describing the probability of a random variable must satisfy the axioms of probability [6], and be non-negative such that the summed probabilities corresponding to the possible values of the random variable must equal 1.0. It follows that if  $F_X(x)$  is a cumulative distribution function, then it must have the following properties:

- (1)  $F_X(-\infty) = 0$ ;  $F_X(+\infty) = 1.0$
- (2)  $F_X(x) \geq 0$ ; and is non-decreasing with  $x$
- (3)  $F_X(x)$  is continuous with  $x$

Probability distributions result due to a physical process (i.e. damage in a pavement system due to repeated loads) which encompass specific assumptions and can be affected by limiting factors (i.e. design and

construction specifications). Probability distributions are typically well-known and developed with widely available statistical information and data tables. The following discussion will present different probability distribution functions for use in pavement design.

The normal distribution. This distribution may be the best known and most widely used distribution in reliability analysis. This distribution, known as a Gaussian distribution ( $N(\mu, \sigma)$ ) has a probability density function as:

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right] \quad \text{for } -\infty < x < \infty$$

where

$\mu$  = mean of the distribution  
 $\sigma$  = standard deviation of the distribution

A gaussian distribution with a mean ( $\mu$ ) of 0.0 and a standard deviation of 1.0 is known as the standard normal distribution ( $N(0,1)$ ) with PDF:

$$f_s(s) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)s^2} \quad \text{for } -\infty < s < \infty$$

is illustrated in the figure below:

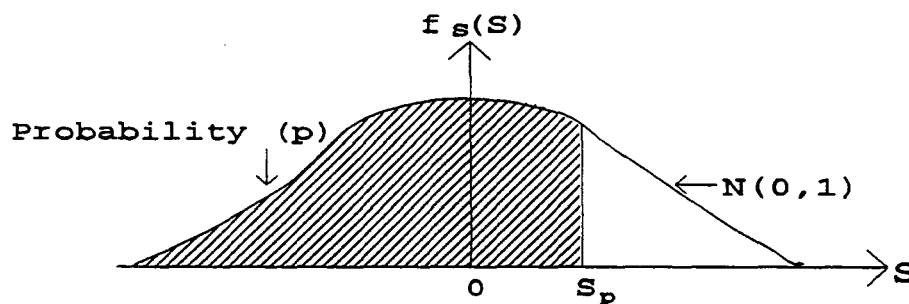


Figure 2. The Standard Normal Density Function [6].

The density function of  $N(0,1)$  is symmetric about zero in which positive values of probability are normally tabulated in available tables. By the virtue of symmetry about zero, the probabilities associated with the negative values of the normal standard variate ( $S$ ) are found from 1 minus the tabulated values:

$$F_S(-s) = 1 - F_S(s)$$

The probabilities ( $p$ ) of other normal distributions  $N(\mu, \sigma)$  can be found for a normal variate ( $X$ ):

$$P(a < X < b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b \exp \left[ -\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right] dx$$

which is the area under a normal curve between the interval from  $a$  to  $b$  as illustrated below [5]:

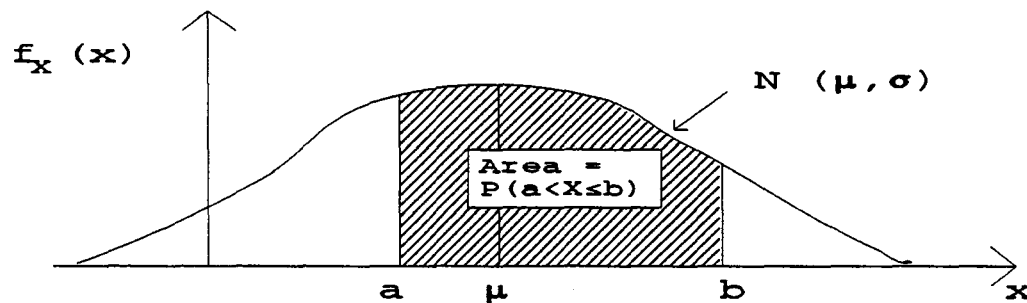


Figure 3. Probability Density Function for  $N(\mu, \sigma)$  [6].

The log-normal distribution. A random variable  $X$  has a log-normal probability distribution of the natural logarithm ( $\ln$ ) of  $X$  is normally distributed. The probability density function of  $X$  is:

$$f_X(x) = \frac{1}{\sqrt{2\pi} \zeta x} \exp \left\{ -\frac{1}{2} \left[ \frac{\ln x - \lambda}{\zeta} \right]^2 \right\} \quad \text{for } 0 \leq x < \infty$$

where

$$\lambda = E(\ln(X))$$

$$\zeta = \sqrt{\text{Var}(\ln X)}$$

The parameter  $x$  is the mean of the distribution of  $\ln X$  and  $\zeta$  is the standard deviation. A log-normal distribution is shown below [5]:

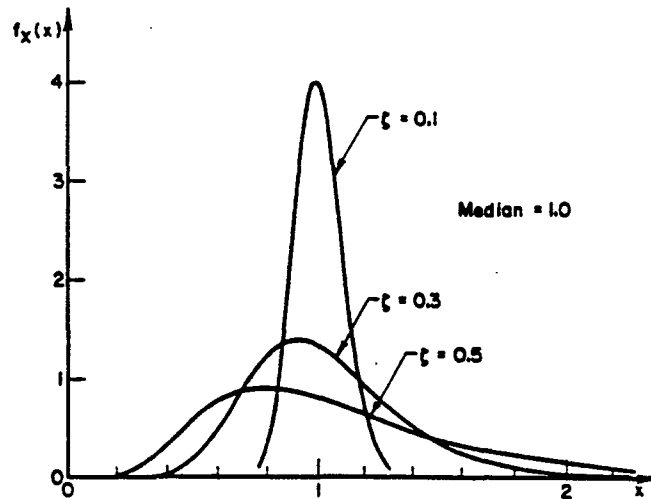


Figure 4. Log-normal density functions. [6]

A log-normal distribution is related to a normal distribution through a logarithmic transformation. The probabilities associated with a log-normal variate ( $X$ ) can be determined using the table of standard normal probabilities. Therefore, the probability of  $X$  between the interval  $a$  to  $b$  is:

$$P(a < X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi} \zeta x} \exp \left[ -\frac{1}{2} \left( \frac{\ln x - \lambda}{\zeta} \right)^2 \right] dx$$

Since the log-normal distribution of the random variable  $X$  displays values which are always positive, this distribution may be useful in applications where the value of the variate is greater than zero. One such application is in the characterization of the distribution of fatigue damage of highway materials [5].

The Weibull distribution. The Weibull distribution typifies the distribution of non-negative random variables occurring in applications regarding life times, waiting times, etc. Other non-negative random distributions include material strengths, particle dimensions, rainfall amounts, radioactive intensities, etc. Exponential or gamma distributions may be used to fit the frequency distribution of this type of random variable; however, the Weibull distribution was introduced to improve the fit of some of these distributions [5, 7].

Experience has indicated that the Weibull distribution can model the probability associated with length of life and endurance data. Fatigue of material components may be related to the "weakest link" interpretation of endurance. If a pavement system is put under stress when the bond between the individual aggregates and the binder may each have its own probabilistic endurance level. As the bond begins to break down under fatigue, the failure process is initiated and the life of the pavement system is related to the minimum fatigue life of any of the bonded aggregates. If the endurance level of any materials exhibit this characteristic, then the Weibull distribution may provide a good approximation of the endurance level distribution.

The probability density function of a random variable  $X$  with a Weibull distribution is of the form:



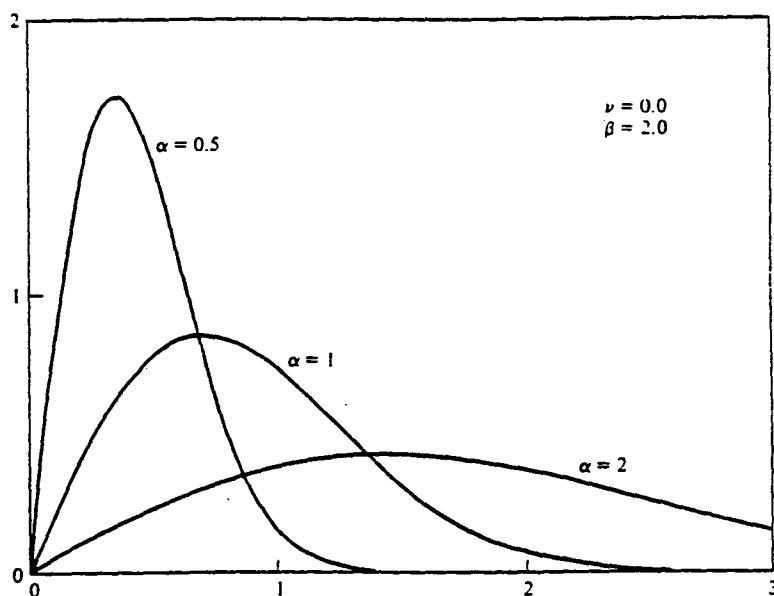


Figure 5. Graphs of the Weibull density functions for  $\nu = 0.0$ ,  $\beta = 2.0$ ,  $\alpha = 0.5, 1.0, 2.0$ . [11].

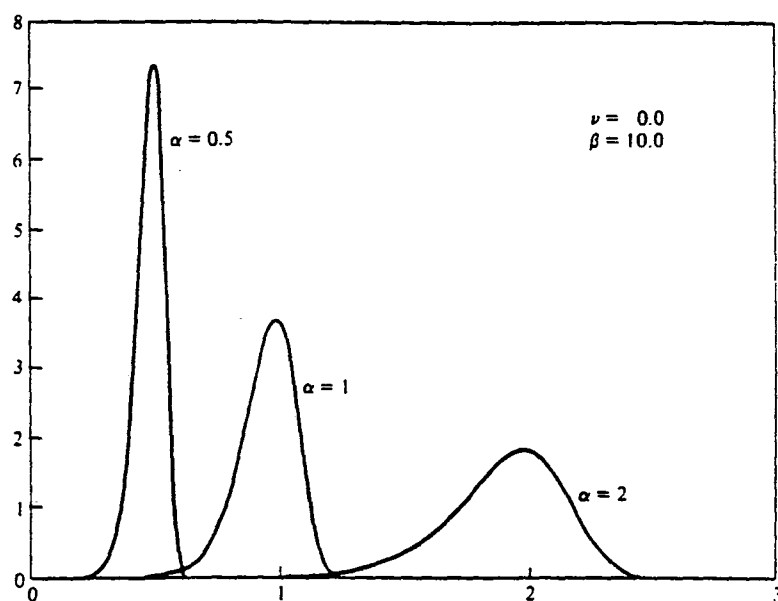


Figure 6. Graphs of the Weibull density functions for  $\nu = 0.0$ ,  $\beta = 10.0$ ,  $\alpha = 0.5, 1.0, 2.0$  [7].

$$f_x(x) = \frac{\beta}{\alpha} \left[ \frac{x-\nu}{\alpha} \right]^{\beta-1} \exp \left[ - \left[ \frac{x-\nu}{\alpha} \right]^\beta \right], \text{ if } x \geq \nu$$

$$0, \quad \text{if } x < \nu$$

Three constants ( $\alpha$ ,  $\beta$ , and  $\nu$ ) are noted as parameters of this distribution. The parameter  $\nu$  represents the smallest possible value of the random variable  $X$ . The parameter  $\beta$  relates to the shape of the density function while  $1/\alpha$  affects the width of the distribution. The different affects of these parameters are shown below.

The Gumbel distribution. This distribution relates to the extremal conditions of a physical process and consequently is referred to as an extreme-value distribution. This type of distribution is a part of an important class of probability distributions involving the extreme values of random variables such as the largest or the smallest values of a random variable. Statistically speaking, these maximum and minimum values represent populations of their own and may be modeled as random variables with probability distributions. Extreme values from observed data are important to many civil engineering applications. One example is in the case of structural safety where high load and low structural resistances are important with regard to the reliability of a structure. When considering extreme conditions, the maxima or minima observations are the only pertinent data.

The largest and smallest values from samples of size  $n$  not only have probability distributions in and of themselves but these distributions can be expected to be related to the distribution of the initial variate or population. Stated in different words, the largest and smallest values from samples of size  $n$  taken from a population  $X$  are considered to be random

variables whose probability distribution is derived from the distribution of the initial variate  $X$ .

To elaborate further, a set of observations  $(x_1, x_2, \dots, x_n)$  is a realization of the sample random variables  $(X_1, X_2, \dots, X_n)$ . In terms of extreme values from a sample size  $n$ , the maximum and minimum of  $(X_1, X_2, \dots, X_n)$  are the random variables:

$$Y_n = \max (X_1, X_2, \dots, X_n)$$

and

$$Y_1 = \min (X_1, X_2, \dots, X_n)$$

If  $Y_n$  is less than a value  $y$ , then all other sample random variables must be less than  $y$ . If  $X_1, X_2, \dots, X_n$  are assumed to be statistically independent and identically distributed as the initial variate  $X$ , then:

$$F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x) = F_X(x)$$

in which

$$\begin{aligned} F_{Y_n}(y) &\equiv p(Y_n \leq y) \\ &\equiv p(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y) \\ &\equiv [F_X(y)]^n \end{aligned}$$

The probability density function for  $Y_n$  is:

$$\begin{aligned} f_{Y_n} &= \frac{\partial F_{Y_n}(y)}{\partial y} \\ &= n [F_X(y)]^{n-1} f_X(y) \end{aligned}$$

For a given  $y$  the probability  $[F_X(y)]^n$  decreases with  $n$  such that the functions  $F_{Y_n}(y)$  and  $f_{Y_n}(y)$  will shift (as illustrated in Figure 7 for an exponential initial distribution  $f_X(x) = e^{-x}$  [8]) with increasing values of  $n$ .

The distribution function for  $Y_1$  can be found in similar fashion.  $Y_1$  is the smallest value in a sample of size  $n$  ( $X_1, X_2, \dots, X_n$ ) and if  $Y_1$  is larger than  $y$ , then all other values in the same sample must be larger than  $y$ . Instead of a distribution function, a survival function is defined [8]:

$$\begin{aligned} 1 - F_{Y_1}(y) &\equiv p(Y_1 > y) \\ &= p(X_1 > y, X_2 > y, \dots, X_n > y) \\ &= [1 - F_X(y)]^n \end{aligned}$$

The distribution function for  $Y_1$  is:

$$F_{Y_1}(y) = 1 - [1 - F_X(y)]^n$$

and the corresponding density function is:

$$f_{Y_1}(y) = n [1 - F_X(y)]^{n-1} f_X(y)$$

The function shifts to the left with increasing  $n$  for the same  $y$ .

The asymptotic distributions of the extremes have been observed to converge on certain limiting forms for large  $n$ . These forms have been classified by Gumbel [9] with respect to a double exponential form and to two different single exponential forms as Type I, Type II, and Type III, respectively:

Type I: The double exponential form,  $\exp[-e^{-A(n)y}]$

Type II: The exponential form,  $\exp[-A(n)/y^k]$

Type III: The exponential form with upper bound  $\underline{w}$ ,  $\exp[-A(n)(\underline{w}-y)^k]$

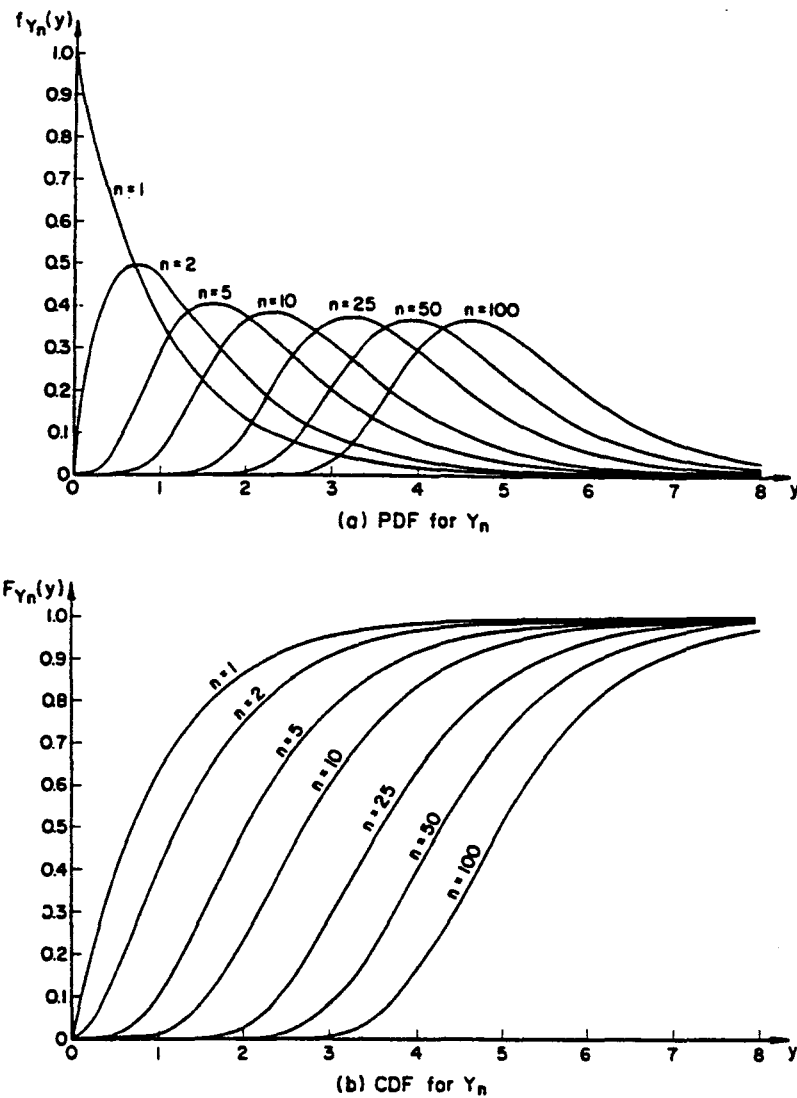


Figure 7. PDF and CDF of the largest value from an exponential initial variate. [14]

This forms also apply to the smallest values [8].

The tail behavior of the initial distribution (in the direction of the extreme) controls to a large extent the convergence of the extreme values of a random variable with respect to a particular limiting form. Specifically, the extreme value from an initial distribution with an

exponentially decaying tail in the direction of the extreme will converge to the Type I limiting form. The extreme value of an initial variate that decays with a polynomial tail will converge to the Type II asymptotic form. The Type III asymptotic form is characteristic of a distribution where the extreme value is limited. In other words, the largest value has a finite upper bound or the smallest value has a finite lower bound.

The Type I Asymptotic Form. The cumulative distribution function (CDF) of the Type I asymptotic form for the distribution of the largest value is:

$$F_{X_n}(x) = \exp[-e^{-\alpha_n(x-U_n)}]$$

where  $U_n$  and  $\alpha_n$  are location and scale parameters, respectively.

- $U_n$  = The characteristic largest value of the initial variate  $X$ .
- $\alpha_n$  = An inverse measure of the dispersion of  $X_n$ .

The corresponding probability density function (PDF) is:

$$f_{X_n}(x) = \alpha_n e^{-\alpha_n(x-U_n)} \exp[-e^{-\alpha_n(x-U_n)}]$$

Similar functions (CDF and PDF) are defined for the smallest value from an initial variate  $X$  [8].

The Type II Asymptotic Form. For the largest value from an initial distribution with a polynomial tail in the direction of the extreme) the asymptotic CDF is:

$$F_{X_n}(x) = \exp\left[-\left(\frac{v_n}{x}\right)^k\right] \quad (14)$$

and the PDF is:

$$f_{X_n}(x) = \frac{k}{v_n} \left(\frac{v_n}{x}\right)^{k+1} \exp\left[-\left(\frac{v_n}{x}\right)^k\right]$$

where

$v_n$  = The characteristic largest value of the initial variate  $X$   
 $k$  = The shape parameter ( $1/k$  is a measure of dispersion).

$v_n$  is also equal to the most probable value of  $X_n$  and similar expressions are defined for the smallest value.

The Type III Asymptotic Form. The Type III asymptotic form represents the limiting distributions that have a finite upper or lower bound. Therefore,  $F_X(\underline{w}) = 1.0$  in the case of the largest value and  $F_X(\epsilon) = 0$  in the case of the smallest value where  $\underline{w}$  is the upperbound value and  $\epsilon$  is the lower bound value. For the largest value, the CDF is:

$$F_{X_n}(x) = \exp \left[ - \left( \frac{\underline{w} - x}{\underline{w} - w_n} \right)^k \right]$$

and the PDF is:

$$f_{X_n}(x) = \frac{k}{\underline{w} - w_n} \left( \frac{\underline{w} - x}{\underline{w} - w_n} \right)^{k-1} \exp \left[ - \left( \frac{\underline{w} - x}{\underline{w} - w_n} \right)^k \right]$$

where  $w_n$  is the characteristic largest value of  $X$  and  $k$  is the shape parameter ( $\underline{w}$  is the upper bound). The parameter  $w_n$  is equal to the modal or most probable value of  $X_n$ . The Type III asymptotic distribution of the smallest value ( $\epsilon$ ) is similar to the Weibull distribution developed in connection with the fatigue behavior of materials.

### Variable Mean and Variances

The normal distribution. The mean associated with a normal distribution is the expected or central value of the random variable as previously described in Chapter 3:

$$\mu_x = E(X)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp \left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right] dx$$

and the variance is found as:

$$\text{Var}(x) = E(X^2) - \mu_x$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp \left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right] dx - \mu_x$$

The log-normal distribution. The parameters associated with a log-normal distribution are  $\lambda$  and  $\zeta$ . These parameters are related to the mean,  $\mu$ , and the variance,  $\sigma^2$ , of the variate. The mean for this distribution is:

$$\mu_1 = E(X) = E(e^Y); Y = \ln X$$

$$= \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^y \exp \left[ -\frac{1}{2} \left( \frac{y - \lambda}{\zeta} \right)^2 \right] dy$$

$$= \left[ \frac{1}{2\pi\zeta} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left( \frac{y - \lambda + \zeta}{\zeta} \right)^2 \right\} dy \right] \exp \left[ \lambda + \frac{1}{2}\zeta^2 \right]$$

Since the quantity within the brackets is the total unit area under the Gaussian density function  $N(\lambda + \zeta^2, \zeta)$ , then:

$$\mu = \exp \left( \lambda + \frac{1}{2}\zeta^2 \right)$$

The variance of  $X$  is a function  $E(X^2)$ :



$$\begin{aligned}
E(X^2) &= \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} e^{2y} \exp \left[ -\frac{1}{2} \left( \frac{y-\lambda}{\zeta} \right)^2 \right] dy \\
&= \left[ \frac{1}{\sqrt{2\pi}\zeta} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left( \frac{y - (\lambda + 2\zeta^2)}{\zeta} \right)^2 \right\} dy \right] \exp[2(\lambda + \zeta^2)] \\
&= \exp[1(\lambda + \zeta^2)]
\end{aligned}$$

As shown previously:

$$\begin{aligned}
\text{Var}(X) &= \exp[2(\lambda + \zeta^2)] - \exp[2(\lambda + \frac{1}{2}\zeta^2)] \\
&= \mu_X(e^{\zeta^2} - 1)
\end{aligned}$$

It can be shown that the median of a log-normal variate is always less than its mean value [5].

The Weibull distribution. It should be noted that if the Weibull parameters  $\beta$  and  $\nu$  are equal to 1 and 0, respectively, that the Weibull distribution is the same as an exponential distribution with parameter  $\theta$  ( $\theta=\frac{1}{\beta}$ ) equal to 1. If the random variable  $X$  has a Weibull distribution with parameter  $\beta$ ,  $\alpha$ , and  $\nu$  and if random variable  $Y$  has an exponential distribution with parameter  $\theta=1$  (where  $Y = [(X-\nu)/\alpha]^\beta$ ) then  $X$  and  $\alpha Y^{1/\beta} + \nu$  have the same distribution, mean, and variance:

$$\begin{aligned}
\mu_X &= E(\alpha Y^{1/\beta} + \nu) = \alpha E(Y^{1/\beta}) + \nu \\
E(X^2) &= E(\alpha Y^{1/\beta} + \nu)^2 = \alpha^2 E(Y^{2/\beta}) + 2\nu\alpha E(Y^{1/\beta}) + \nu^2
\end{aligned}$$

where

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r (\theta e^{-\theta x}) dx = \frac{1}{\theta^r} \int_0^\infty v^r e^{-v} dv \\
&= \frac{\Gamma(r+1)}{\theta^r} \text{ (for exponential distribution)}
\end{aligned}$$

where

$$\begin{aligned}\Gamma &= \text{Gamma function} \\ r &= 1/B, 2/B, \dots, n/B\end{aligned}$$

therefore

$$E(Y^{1/B}) = \Gamma\left[1 + \frac{1}{\beta}\right]$$

$$E(Y^{2/B}) = \Gamma\left[1 + \frac{2}{\beta}\right]$$

and

$$\begin{aligned}\mu_X &= \alpha \Gamma\left[1 + \frac{1}{\beta}\right] + \nu \\ \sigma_X^2 - E(X^2) - (\mu_X)^2 &= \alpha^2 \left\{ \Gamma\left[1 + \frac{2}{\beta}\right] - \left[ \Gamma\left[1 + \frac{1}{\beta}\right] \right]^2 \right\}\end{aligned}$$

The Weibull distribution depends on  $B$  and  $\alpha$  and not on  $\nu$ . The parameter  $\nu$  is a location parameter which does not affect the shape of density function.

The Gumbel distribution. The expected value and variance of a random variable  $X$  can be derived from the higher-order moments of the variable, since the probability distribution would be completed if all of the moments of a random variable were known. Therefore, a function ((known as a moment-generating function) by which all of the moments can be generated is a method of providing for the probability distribution of a random variable.

A moment-generating function, denoted as  $G_X(s)$ , of a random variable  $X$  is defined as the expected value of  $e^{sX}$ :

$$G_X(s) \equiv E(e^{sX})$$

where  $s$  is a deterministic variable. The corresponding moment-generating function for PDF  $f_X(x)$  is:

$$G_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

and

$$\frac{dG_X(s)}{ds} \Big|_{s=0} = \int_{-\infty}^{\infty} x f_X(x) dx = E(X)$$

Similarly,

$$\frac{d^2 G_X(s)}{ds^2} \Big|_{s=0} = \int_{-\infty}^{\infty} x^2 f_X(x) dx = E(X^2)$$

which can be used to find  $\text{Var}(X)$ , since:

$$\text{Var}(X) = E(X^2) - \mu_X^2$$

The Type I Asymptotic Form. For the Type I largest value, the standardized extremal variate ( $S$ ) can be defined as:

$$S = \alpha_n (X_n - U_n) \quad (15)$$

in which the moment-generating function of  $S$  is:

$$\begin{aligned} G_S(t) &= E(e^{tS}) \\ &= \int_{-\infty}^{\infty} e^{ts} e^{-s} \exp[-e^{-s}] ds \end{aligned}$$

If  $r = e^{-s}$  and  $ds = -(dr)/r$ , then:

$$\begin{aligned} G_S(t) &= \int_0^{\infty} e^{ts} e^{-r} dr \\ &= \int_0^{\infty} e^{-t \ln r} e^{-r} dr = \int_0^{\infty} r^{-t} e^{-r} dr \\ &= \Gamma(1 - t) \end{aligned}$$

The derivatives of  $G_S(t)$  (evaluated at  $t=0$ ) provide the moments of  $S$ , Gumbel

[15] showed that:

$$E(S) = \frac{d G_S(t)}{dt} = \frac{d\Gamma(1)}{dt} = \gamma = 0.577216\dots(\text{Euler the number})$$

$$\text{Var}(S) = E(S^2) - \mu_S^2$$

$$= \frac{d^2 G_S(t)}{ds^2} - \mu_S^2 = \frac{d^2 \Gamma(1)}{dt^2} - \mu_S^2 = \gamma^2 + \frac{\pi^2}{6} - \gamma^2 = \frac{\pi^2}{6}$$

The moments of a Type I extremal variate may be evaluated from equation 15

$$\left[ S = \alpha_n \left( \frac{X_n - \mu_n}{\sigma_s} \right) = \left( \frac{X_n - \mu_n}{\sigma_{X_n}} \right) \right] :$$

$$X_n = \mu_{X_n} = U_n + \frac{S}{\alpha_n} = \mu_n + \frac{\gamma}{\alpha_n}$$

and

$$\sigma_{X_n}^2 = \frac{1}{\alpha_n^2} \sigma_S^2 = \frac{\pi^2}{6\alpha_n^2}$$

Similar analysis will provide the mean and variance for the smallest value.

The Type II Asymptotic Form. If  $X_n$  has the Type II asymptotic distribution given by equation 14 with parameters  $v_n$  and  $k$ , then the distribution of  $\ln X_n$  will have the Type I asymptotic form with:

$$U_n = \ln v_n$$

$$\alpha_n = k$$

The standardized extremal variate is:

$$\begin{aligned} S &= k (\ln X_n - \ln v_n) \\ &= k \ln \frac{X_n}{v_n} \end{aligned}$$

In terms of moment-generating functions, the  $t^{\text{th}}$  moment of  $X_n$  is:

$$\begin{aligned} E(X_n^t) &= v_n^t E[e^{(t/R)S'}] \\ &= v_n^t G_*\left(\frac{t}{k}\right) = v_n^t \Gamma\left[1 - \frac{t}{k}\right] \end{aligned}$$

for  $t < k$ :

$$X_n = \mu_{X_n} = v_n \Gamma\left[1 - \frac{1}{k}\right]$$

and for  $k \geq 2$ :

$$\begin{aligned} E(X_n^2) &= v_n \Gamma\left[1 - \frac{2}{k}\right] \\ \sigma_{X_n}^2 &= v_n \left[ \Gamma\left[1 - \frac{2}{k}\right] - \Gamma^2\left[1 - \frac{1}{k}\right] \right] \end{aligned}$$

Similar expressions can be found for the smallest value by replacing  $v_n$  by  $v_1$  (the characteristic smallest value of the initial variate  $X$ ).

The Type III Asymptotic Form. The standardized Type III extremal variate for the largest value is:

$$\begin{aligned} S &= \alpha_n (X_n - U_n) \\ &= -k \ln \left( \frac{\underline{w} - X_n}{\underline{w} - w_n} \right) \\ e^{-S/k} &= \frac{\underline{w} - X_n}{\underline{w} - w_n} \\ \underline{w} - X_n &= (\underline{w} - w_n) e^{-S/k} \\ E(\underline{w} - X_n)^t &= (\underline{w} - w_n)^t E(e^{-tS/k}) \\ &= (\underline{w} - w_n)^t G_*(-t/k) \\ &= (\underline{w} - w_n)^t \Gamma(1 + t/k) \end{aligned}$$

which provides for the  $t^{\text{th}}$  moment of  $\underline{w} - X_n$  which leads to the moment of  $X_n$

and the mean of  $X_n = \mu_{X_n}$  :

$$\mu_{X_n} = \underline{w} - (\underline{w} - w_n) \Gamma \left[ 1 + \frac{1}{k} \right]$$

$$E(\underline{w} - X_n)^2 = (\underline{w} - w_n)^2 \Gamma \left[ 1 + \frac{2}{k} \right]$$

thus

$$\sigma_{X_n}^2 = (\underline{w} - w_n)^2 \left[ \Gamma \left[ 1 + \frac{2}{k} \right] - \Gamma^2 \left[ 1 + \frac{1}{k} \right] \right]$$

Similar expressions can be found for the smallest value by replacing  $\underline{w}$  with  $\epsilon$ .

The distributions which have been discussed above are summarized in Tables 1 and 2 and are only a few of the types of distributions which may be selected for engineering design analysis. Other distributions are provided in the tables but are not elaborated.

Table 1. Common Distributions and Their Parameters.

Distribution	Probability density function (PDF) or mass function (PMF)	Parameters	Relation to Mean and Variance
Normal (Gaussian)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ $-\infty < x < \infty$	$\mu, \sigma$	$\text{Var}(X) = \sigma^2$
Lognormal	$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right]$ $x \geq 0$	$\lambda, \zeta$	$E(X) = \exp\left(\lambda + \frac{1}{2}\zeta^2\right)$ $\text{Var}(X) = E^2(X)[e^{\zeta^2} - 1]$
Weibull	$f_X(x) = \frac{\beta}{\alpha} \left(\frac{x-v}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-v}{\alpha}\right)^\beta\right], \text{ if } x \geq v$ $0, \text{ if } x < v$	$\alpha, \beta, v$	$E(X) = \alpha E(Y^{1/\beta}) + v$ $Y = \left(\frac{x-v}{\alpha}\right)^\beta$ $\text{Var}(X) = \alpha^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\}$
Exponential	$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$	$\lambda$	$\text{Var}(X) = 1/\lambda$ $E(X) = 1/\lambda$
Gamma	$f_X(x) = \frac{v (vx)^{k-1} e^{-vx}}{\Gamma(k)} \quad x \geq 0$	$v, k$	$\text{Var}(X) = k/v^2$ $E(X) = \mu$
Rayleigh	$f_X(x) = \frac{x}{\alpha^2} \exp\left[-\frac{1}{2}\left(\frac{x}{\alpha}\right)^2\right] \quad x \geq 0$	$\alpha$	$\text{Var}(X) = \left(2 - \frac{\pi}{2}\right) \alpha^2$ $E(X) = (\alpha + b)/2$
Binomial	$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2, \dots, n$	$p$	$E(X) = np$ $\text{Var}(X) = np(1-p)$
Geometric	$p_X(x) = p(1-p)^{x-1}$ $x = 0, 1, 2, \dots$	$p$	$E(X) = 1/p$ $\text{Var}(X) = (1-p)/p^2$ $E(X) = vt$
Poisson	$p_X(x) = \frac{(vt)^x}{x!} e^{-vt}$ $x = 0, 1, 2, \dots$	$v$	$\text{Var}(X) = vt$
Uniform	$f_X(x) = \frac{1}{b-a} \quad a < x < b$	$a, b$	$\text{Var}(X) = \frac{1}{12} (b-a)^2$
Triangular	$f_X(x) = \frac{2}{b-a} \left(\frac{x-a}{u-a}\right) \quad a \leq x \leq u$ $= \frac{2}{b-a} \left(\frac{b-x}{b-u}\right) \quad u \leq x \leq b$	$a, b, u$	$E(X) = \frac{1}{3} (a+b+u)$ $\text{Var}(X) = \frac{1}{18} (a^2+b^2+u^2 - ab - au - bu)$
Beta	$f_X(x) = \frac{1}{B(q,r)} \frac{(x-a)^{q-1} (b-x)^{r-1}}{(b-a)^{q+r-1}}$ $a \leq x \leq b$	$a, b, q, r$	$E(X) = a + \frac{q}{q+r} (b-a)$ $\text{Var}(X) = \frac{qr}{(q+r)^2 (q+r+1)} (b-a)^2$

Table 2. The Three Types of Asymptotic Extremal Distributions.

Asymptotic Type	Tail Characteristics of Initial Variate	Extreme	Cumulative Distribution Function	Mean Value	Standard Deviation	Standard Extremal Variate, S
I	Exponential	Largest	$\exp[-e^{-\alpha_n(x_n - u_n)}]$	$u_n + \frac{0.577}{\alpha_n}$	$\frac{\pi}{\sqrt{6}\alpha_n}$	$\alpha_n (X_n - u_n)$
		Smallest	$\exp[-e^{\alpha_1(x_1 - u_1)}]$	$u_1 - \frac{0.577}{\alpha_1}$	$\frac{\pi}{\sqrt{6}\alpha_1}$	$-\alpha_1 (X_1 - u_1)$
II	Polynomial	Largest	$\exp\left[1 - \left(\frac{v_n}{y_n}\right)^k\right]$	$v_n \Gamma\left(1 - \frac{1}{k}\right)$	$v_n \left[\Gamma\left(1 - \frac{2}{k}\right) - \Gamma^2\left(1 - \frac{1}{k}\right)\right]^{1/2}$	$k \ln \frac{y_n}{v_n}$
		Smallest	$\exp\left[-\left(\frac{v_1}{y_1}\right)^k\right]$	$v_1 \Gamma\left(1 - \frac{1}{k}\right)$	$v_1 \left[\Gamma\left(1 - \frac{2}{k}\right) - \Gamma^2\left(1 - \frac{1}{k}\right)\right]^{1/2}$	$k \ln \frac{v_1}{y_1}$
III	Bounded (in direction of extreme)	Largest	$\exp\left[1 - \left(\frac{w - z_n}{w - w_n}\right)^k\right]$	$w - (w - w_n) \Gamma\left(1 + \frac{1}{k}\right)$	$(w - w_n) \left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]^{1/2}$	$-k \ln \left(\frac{w - z_n}{w - w_n}\right)$
		Smallest	$\exp\left[-\left(\frac{z_1 - \epsilon}{w_1 - \epsilon}\right)^k\right]$	$\epsilon + (w_1 - \epsilon) \Gamma\left(1 + \frac{1}{k}\right)$	$(w_1 - \epsilon) \left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]^{1/2}$	$-k \ln \left(\frac{z_1 - \epsilon}{w_1 - \epsilon}\right)$



### APPENDIX III

#### PAVEMENT CRACKING FOR EQUIVALENT LOADS

$$\text{Fatigue Damage (D)} = D(n, N_f) = \sum_i \frac{n}{N_{fi}}$$

where

$n$  = actual traffic equivalent load applications  
 $N_{fi}$  = number of equivalent load applications to reach an unacceptable level of distress

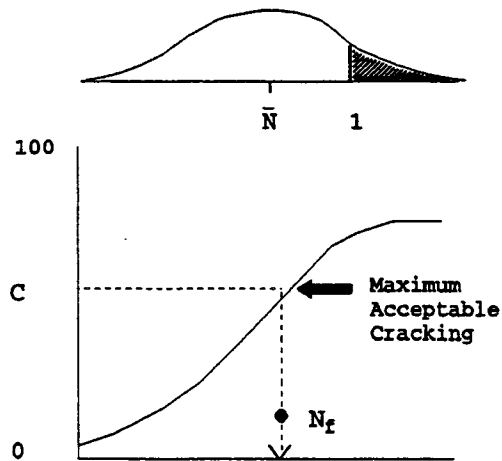
$$\begin{aligned} \text{Fatigue Cracking (C)} &= 100 \text{Prob } [D > 1] \text{ (percent)} \\ &= 100 \int_1^{\infty} p(x) dx \end{aligned}$$

where

$p(x)$  = probability density function for damage (for normal, log normal, and Weibull distributions)

$$x = \frac{(D - \bar{D})}{\sigma_D} = \frac{n - \bar{n}}{N_{fi}} \left[ \frac{1}{\sigma_D} \right]; \sigma_D = \sqrt{\text{Var } D}$$

$$C = 100 \int_1^{\infty} p \left[ \left( \frac{n - \bar{n}}{N_{fi}} \right) \frac{1}{\sqrt{\text{Var } D}} \right] dx$$



Variance of fatigue damage

$$\text{Var} [D] = \left( \frac{\partial D}{\partial n} \right)^2 \text{Var} (n) + \left( \frac{\partial D}{\partial N_f} \right)^2 \text{Var} (N_f)$$

where

$$\frac{\partial D}{\partial n} = \frac{1}{N_f}$$

$$\frac{\partial D}{\partial N_f} = \frac{-n}{N_f^2}$$

$\text{Var} (n)$  = variance of load applications  
 $\text{Var} (N_f)$  = variance of load applications to failure

$$\text{Var } [D] = \left( \frac{1}{N_f} \right)^2 \text{Var}(n) + \left( \frac{n^2}{N_f^4} \right) \text{Var}(N_f)$$

$$\sigma_D = \frac{1}{N_f} \left[ \text{Var}(n) + D^2 \text{Var}(N_f) \right]^{\frac{1}{2}}$$

$$= \frac{\sigma_n}{N_f} \left\{ 1 + D^2 \frac{\text{Var}(N_f)}{\text{Var}(n)} \right\}^{\frac{1}{2}} \quad cv^2 = \frac{\sigma^2}{x^2}$$

$$= \frac{\sigma_n}{N_f} \left\{ 1 + \frac{cv^2 N_f}{cv^2 n} \right\}^{\frac{1}{2}}$$

$$x = \frac{n - \bar{n}}{N_f} \frac{N_f}{\sigma_n} \left\{ 1 + \frac{cv^2 N_f}{cv^2 n} \right\}^{\frac{1}{2}}$$

$$= \frac{n - \bar{n}}{\sigma_n \left\{ 1 + \frac{cv^2 N_f}{cv^2 n} \right\}^{\frac{1}{2}}}$$

$$dx = \frac{dn}{\sigma_n \left\{ 1 + \frac{cv^2 N_f}{cv^2 n} \right\}^{\frac{1}{2}}}$$

